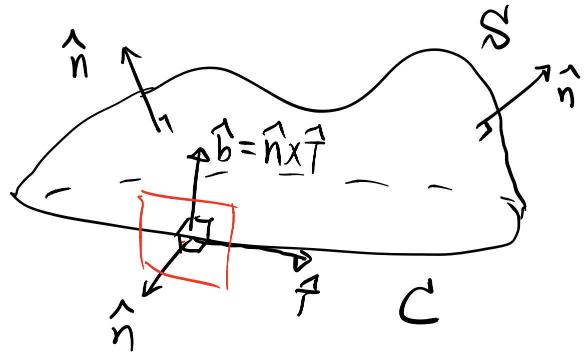
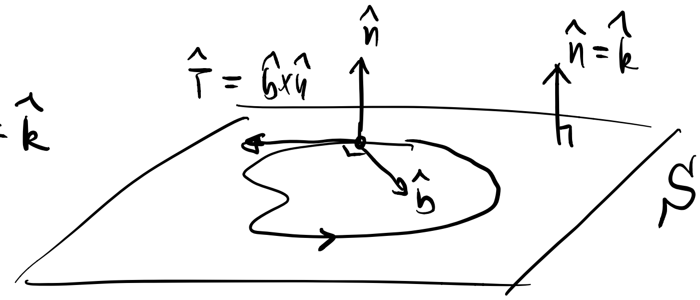
eg60

(1)



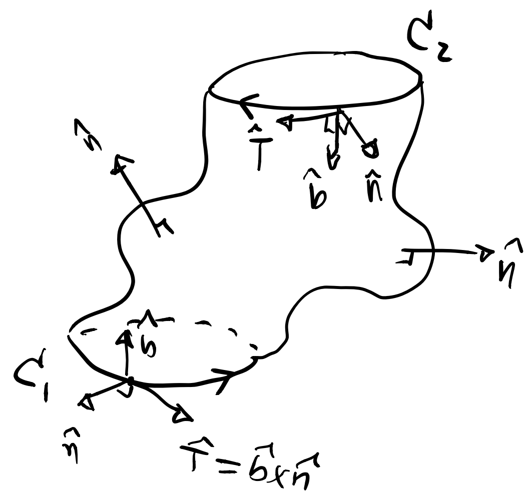
(upward normal)

(2) If $S \subset \mathbb{R}^2$ with $\hat{n} = \hat{k}$



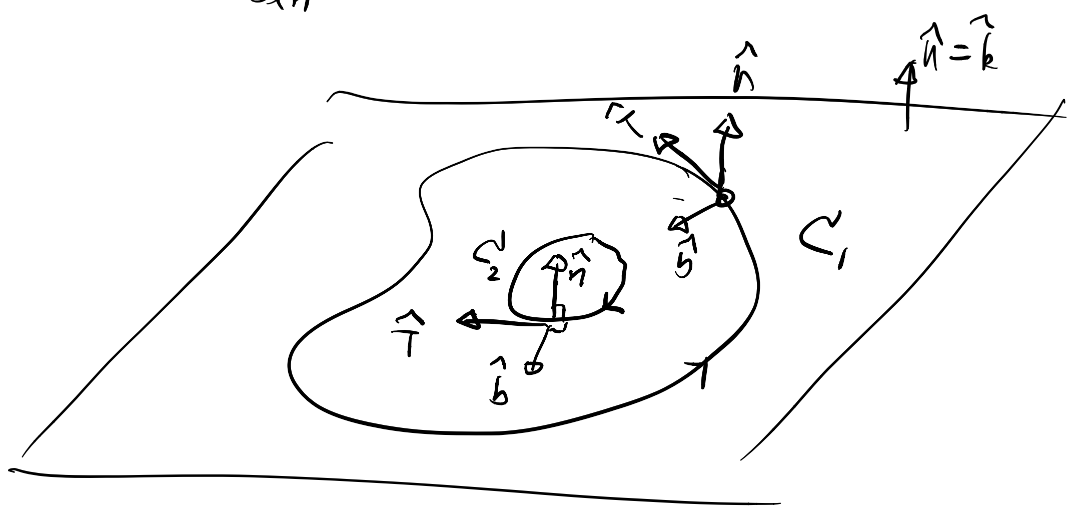
same as the anti-clockwise orientation in the plane.

(3)

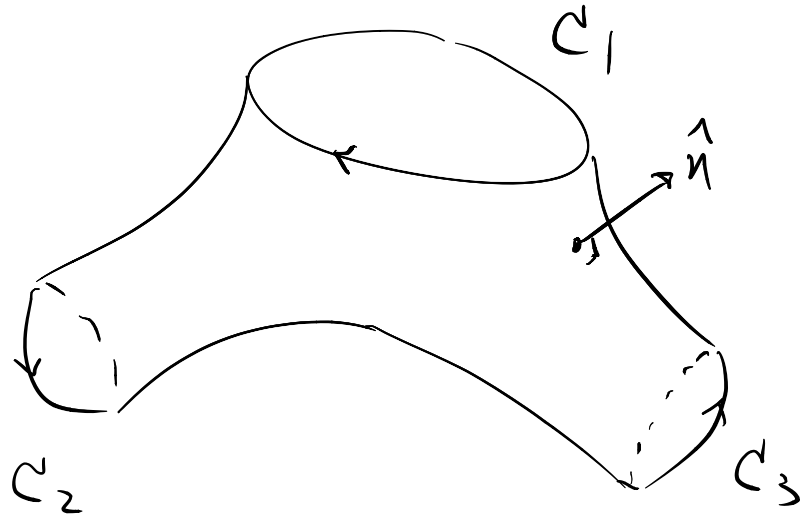


$C = C_1 \cup C_2$

(4)



(5)

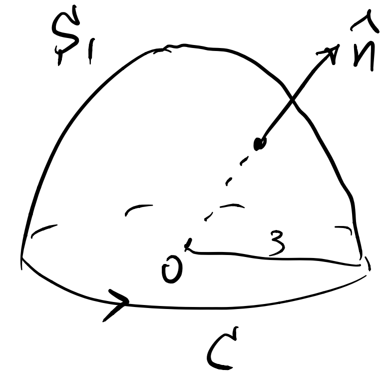


(Ex!)

<u>eg 61</u>

(a) $S_1: x^2+y^2+z^2=9, \ z \geq 0$ with upward normal,

boundary $C: x^2+y^2=9, \ z=0$



Let $\vec{F} = y\hat{i} - x\hat{j}$

<u>Verifying Stokes' Thm</u>:

$C: \vec{r}(t) = (3\cos t, 3\sin t, 0) \ , \ 0 \leq t \leq 2\pi$

$= 3\cos t \, \hat{i} + 3\sin t \, \hat{j}$

(has the correct orientation)

$d\vec{r} = (-3\sin t \, \hat{i} + 3\cos t \, \hat{j}) dt$

Along C, $\vec{F}(\vec{r}(t)) = y\hat{i} - x\hat{j} = 3\sin t \, \hat{i} - 3\cos t \, \hat{j}$

$\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (3\sin t \, \hat{i} - 3\cos t \, \hat{j}) \cdot (-3\sin t \, \hat{i} + 3\cos t \, \hat{j}) dt$

$= -18\pi$ (check!)

For the surface integral

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = -2\hat{k} \quad \text{(check!)}$$

Since S_1 is a hemisphere centered at origin

$$\hat{n} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{1}{3}(x\hat{i}+y\hat{j}+z\hat{k}) \text{ on } S_1$$

The surface S_1 can be regarded as level surface given by

$$g(x,y,z) = x^2+y^2+z^2 = 9$$

Note $\vec{\nabla}g = (2x, 2y, 2z)$

Since $z>0$ (except the boundary) on S_1

$$\frac{\partial g}{\partial z} = 2z \neq 0$$

Hence $$d\sigma = \frac{|\vec{\nabla}g|}{\left|\frac{\partial g}{\partial z}\right|}dxdy = \frac{\sqrt{4x^2+4y^2+4z^2}}{|2z|}dxdy$$

$$= \frac{3}{|z|}dxdy = \frac{3}{z}dxdy$$

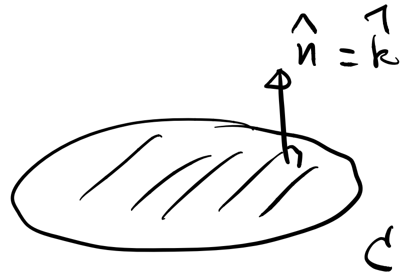
Therefore $\iint_{S_1} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma$

$$= \iint_{x^2+y^2\leq 9} (-2\hat{k})\cdot\frac{1}{3}(x\hat{i}+y\hat{j}+z\hat{k})\,\frac{3}{z}\,dx\,dy$$

$$= \iint_{x^2+y^2\leq 9} (-2)\,dx\,dy = -18\pi \quad \text{(check!)}$$

(b) $S_2 : x^2+y^2 \leq 9,\ z=0$

(new surface, same C and same $\vec{F}$)



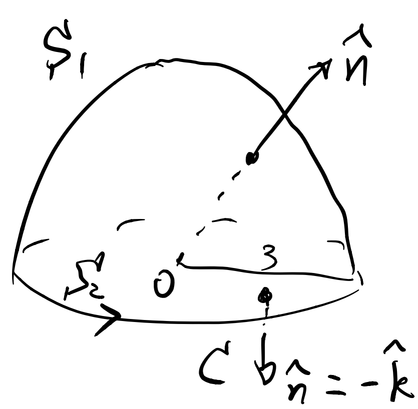
$$\iint_{S_2} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma = \iint_{x^2+y^2\leq 9} (-2\hat{k})\cdot\hat{k}\, d\sigma$$

$$= -2\iint_{x^2+y^2\leq 9} d\sigma = -18\pi \quad \text{(check!)}$$

(c) $\vec{F} = y\hat{i} - x\hat{j}$ (same $\vec{F}$)

$S_3 = S_1 \cup S_2$

S_3 has no boundary



and encloses a solid region

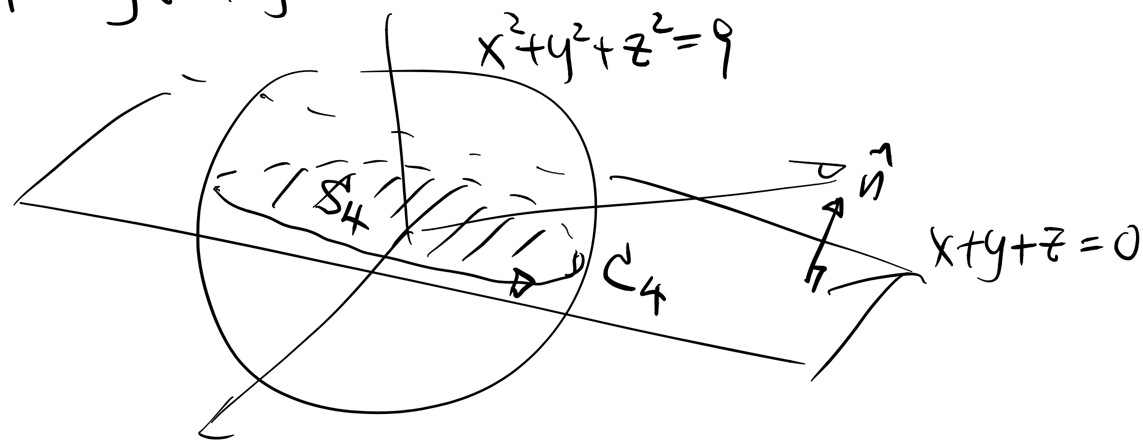
Suppose $\hat{n}$ = outward normal (of the solid)

$$\iint_{S_3} (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, d\sigma = \iint_{S_1} (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, d\sigma + \iint_{S_2} (\vec{\nabla} \times \vec{F}) \cdot (-\hat{k}) \, d\sigma$$

$$= \iint_{S_1} (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, d\sigma - \iint_{S_2} (\vec{\nabla} \times \vec{F}) \cdot \hat{k} \, d\sigma$$

$$= -18\pi - (-18\pi) = 0 .$$

(d) Same $\vec{F} = y\hat{i} - x\hat{j}$ (new surface, new C)



$$S_4 = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 9 , \; x+y+z=0 \}$$

Applying Stokes' Thm

$$\oint_{C_4} \vec{F} \cdot d\vec{r} = \iint_{S_4} (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, d\sigma$$

$$= \iint_{S_4} (-2\hat{k}) \cdot \frac{1}{\sqrt{3}} (\hat{i} + \hat{j} + \hat{k}) \, d\sigma$$

$$= -\frac{2}{\sqrt{3}} \iint_{S_4} d\sigma$$

$$= -\frac{2}{\sqrt{3}} \text{Area}(S_4)$$

$$= -\frac{2}{\sqrt{3}}(\pi \cdot 3^2) = -\frac{18\pi}{\sqrt{3}}$$

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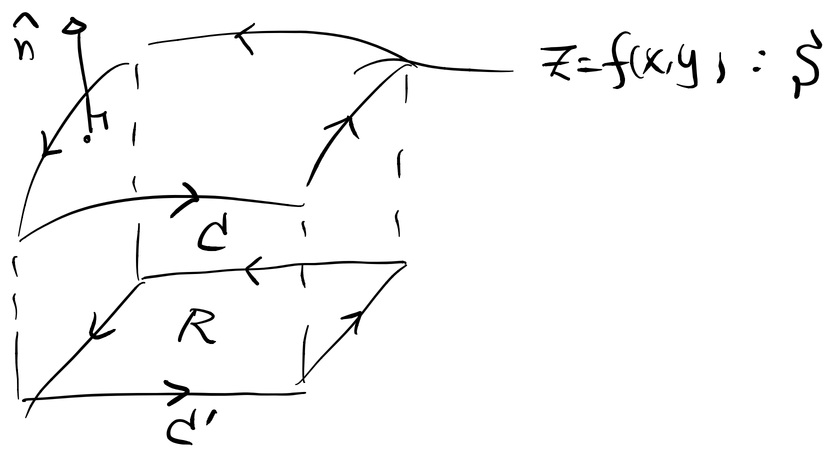
Proof of Stokes' Thm

<u>Special case</u>: S is a graph given by

$z = f(x,y)$ over a region R with <u>upward</u> normal



Assume C is the boundary of S, and C' is the boundary of R (anti-clockwisely oriented wrt the normal of S and the plane respectively)

Parametrize the graph as

$$\vec{r}(x,y) = x\hat{i} + y\hat{j} + f(x,y)\hat{k} \qquad (x,y) \in R$$

Then $\begin{cases} \vec{r}_x = \hat{i} + \dfrac{\partial f}{\partial x}\hat{k} \\ \vec{r}_y = \hat{j} + \dfrac{\partial f}{\partial y}\hat{k} \end{cases}$

$$\Rightarrow \quad \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}$$

(upward)

hence $\hat{n} = \dfrac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|}$ is the upward normal of S',

and $\quad d\sigma = |\vec{r}_x \times \vec{r}_y|\, dx\, dy = |\vec{r}_x \times \vec{r}_y|\, dA$

area element of R.

Let $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ be the C^1 vector field

Then $$\iint_{S'} (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, d\sigma = \iint_R (\vec{\nabla} \times \vec{F})(\vec{r}(x,y)) \cdot \frac{\vec{r}_x \times \vec{r}_y}{\cancel{|\vec{r}_x \times \vec{r}_y|}} \cancel{|\vec{r}_x \times \vec{r}_y|}\, dA$$

$$= \iint_R \left[(L_y - N_z)\hat{i} + (M_z - L_x)\hat{j} + (N_x - M_y)\hat{k}\right] \cdot \left[-\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}\right] dA$$

$$= \iint_R \left[-f_x(L_y - N_z) - f_y(M_z - L_x) + (N_x - M_y)\right] dx\, dy$$

For the line integral

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C M\,dx + N\,dy + L\,dz$$

(thinking of $d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$)

$$= \oint_{C'} M\,dx + N\,dy + L\,df \qquad z = f(x,y)$$

$$= \oint_{C'} M\,dx + N\,dy + L\,(f_x\,dx + f_y\,dy)$$

$$= \oint_{C'} (M + Lf_x)\,dx + (N + Lf_y)\,dy$$

Remark: If C' is parametrized by

$$\vec{\gamma}(t) = (x(t), y(t)) \quad \text{for } a \le t \le b$$

then C is parametrized by

$$\vec{r}(t) = (x(t), y(t), f(x(t), y(t)))$$

$$= x(t)\,\hat{i} + y(t)\,\hat{j} + f(x(t), y(t))\,\hat{k}$$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = \int_a^b \left[M(\vec{r}(t))\,x'(t) + N(\vec{r}(t))\,y'(t) + L(\vec{r}(t))\,\frac{d}{dt}\big(f(x(t), y(t))\big) \right] dt$$

$$= \int_a^b \left[Mx' + Ny' + L\,(f_x x' + f_y y') \right] dt$$

$$= \int_a^b \left[(M + L f_x) x' + (N + L f_y) y' \right] dt$$

$$= \oint_{C'} (M + L f_x)\,dx + (N + L f_y)\,dy$$

Then by Green's Thm

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_{C'} (M + L f_x)\,dx + (N + L f_y)\,dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}(N + L f_y) - \frac{\partial}{\partial y}(M + L f_x) \right] dA$$

$$= \iint_R \left\{ \begin{array}{l} \frac{\partial}{\partial x}\left[N(x,y,f(x,y)) + L(x,y,f(x,y)) f_y(x,y) \right] \\ - \frac{\partial}{\partial y}\left[M(x,y,f(x,y)) + L(x,y,f(x,y)) f_x(x,y) \right] \end{array} \right\} dA$$

$$= \iint_R \left[\begin{array}{l} (N_x + N_z f_x) + (L_x + \cancel{L_z f_x}) f_y + \cancel{L f_{yx}} \\ - (M_y + M_z f_y) - (L_y + \cancel{L_z f_y}) f_x - \cancel{L f_{xy}} \end{array} \right] dA$$

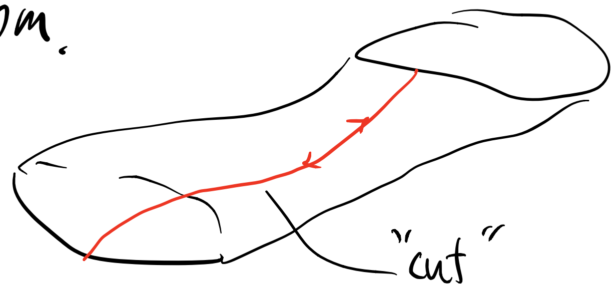
$$= \iint_R \left[-f_x (L_y - N_z) - f_y (M_z - L_x) + (N_x - M_y) \right] dA$$

$$= \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, d\sigma .$$

This completes the case for C^2 graph.

<u>General case</u>: Divides S into finitely many pieces which are graphs (in certain projection)

(This includes S with many boundary components as in Green's Thm.



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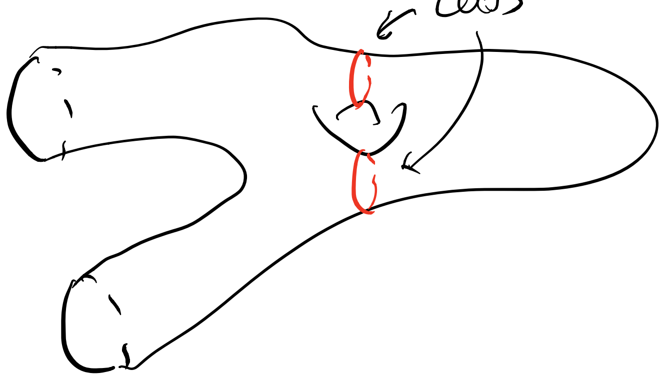
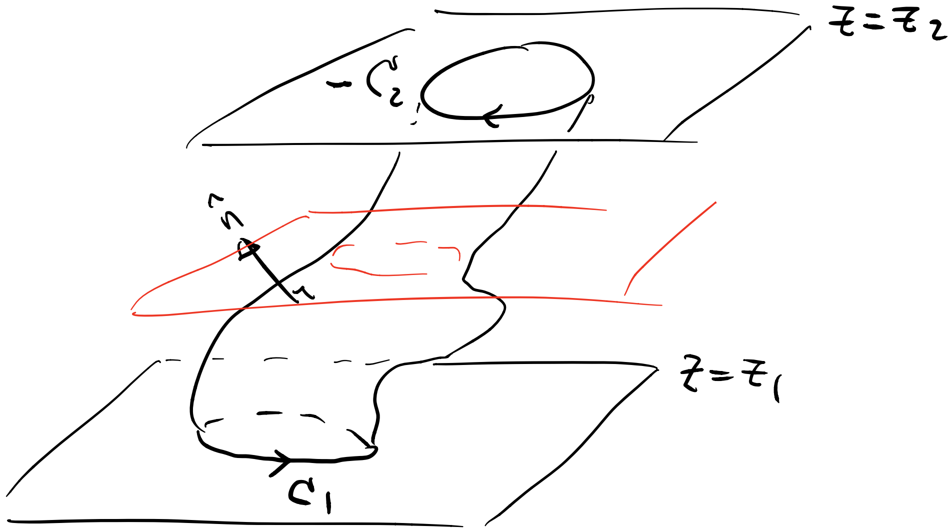
<u>Note</u>: Stokes' Thm applies to surfaces like the following:



<u>eg 62</u>: Let $\vec{F}$ be a vector field such that $\underline{\vec{\nabla}\times\vec{F}=0}$ and defined on a region containing the surface S with unit normal vector field $\vec{n}$ as in the figure:



The boundary C of S has 2 components C_1 and C_2 at the level $z=z_1$, and $z=z_2$ respectively.

If both C_1 and C_2 oriented anticlockwisely with respect to the "horizontal planes"

Then when C oriented with respect to $\hat{n}$, then

$$C = C_1 - C_2 \, .$$

And Stokes' Thm $\Rightarrow$

$$0 = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma = \oint_C \vec{F}\cdot d\vec{r}$$

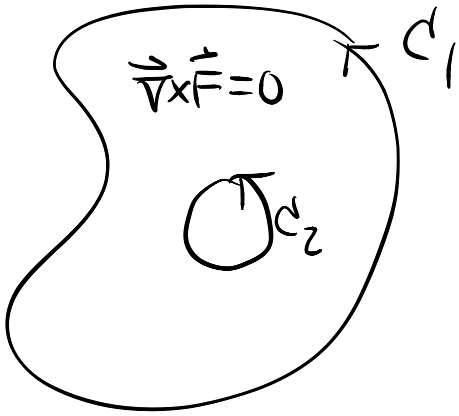
$\nwarrow$ oriented wrt $\vec{n}$

$$= \oint_{C_1} \vec{F}\cdot d\vec{r} - \oint_{C_2} \vec{F}\cdot d\vec{r}$$

oriented wrt "plane".

$$\Rightarrow \quad \oint_{C_1} \vec{F}\cdot d\vec{r} = \oint_{C_2} \vec{F}\cdot d\vec{r}$$

Compare this with Green's Thm on plane region with one hole



$$\Rightarrow \quad \oint_{C_1} \vec{F}\cdot d\vec{r} = \oint_{C_2} \vec{F}\cdot d\vec{r} \qquad \text{(check!)}$$

anti-clockwisely wrt "plane".