Recall Green's Thm

- Normal Form

$$\oint_C \vec{F}\cdot\hat{n}\, ds = \boxed{\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy}$$

- Tangential Form

$$\oint \vec{F}\cdot\hat{T}\, ds = \boxed{\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$

eg 48 Verify both form of Green's Thm for

$\vec{F}(x,y) = (x-y)\hat{i} + x\hat{j}$ on $\Omega = \mathbb{R}^2$ (is C^∞)

C = unit circle : $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $0 \le t \le 2\pi$

Then R = region enclosed by C = $\{x^2+y^2<1\}$ the unit disc

(We also write $C = \partial R$ boundary of R)

Soln $M = x-y$, $N = x$ (in this case)

$\frac{\partial M}{\partial x} = 1$, $\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = 1$, $\frac{\partial N}{\partial y} = 0$

On C , $x = \cos t$, $y = \sin t$ for $0 \le t \le 2\pi$

Normal Form

$$\text{L.H.S.} = \oint_C M\,dy - N\,dx$$

$$= \int_0^{2\pi} \left[(\cos t - \sin t)\cos t - \cos t(-\sin t)\right] dt$$

$$= \int_0^{2\pi} \cos^2 t = \pi$$

$$\text{R.H.S.} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R (1+0)\, dA = \pi$$

Tangential Form

$$\text{L.H.S.} = \oint_C M\,dx + N\,dy$$

$$= \int_0^{2\pi} [(\cos t - \sin t)(-\sin t) + \cos t \cos t]\,dt$$

$$= \int_0^{2\pi} (-\cos t \sin t + 1)\,dt = 2\pi$$

$$\text{R.H.S.} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R (1-(-1))\,dA = 2\pi \quad \#$$

(Note: this example shows that even the 2 forms are equivalent, the values involved may differ.)

Pf of Green's Thm (tangential form)

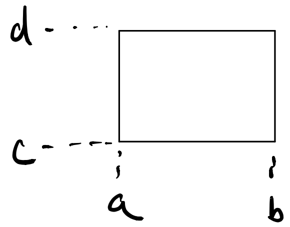
Recall: A region R is of special type:

type (1): If $R = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$
for some continuous function $g_1(x)$ & $g_2(x)$.

type (2): If $R = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$
for some continuous function $h_1(x)$ & $h_2(x)$.

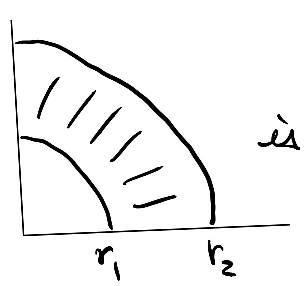
Now: If R is both type (1) & type (2), it said to be simple.

eg 48 (i)



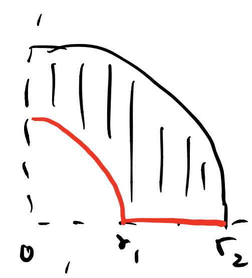
rectangle is simple

(ii)

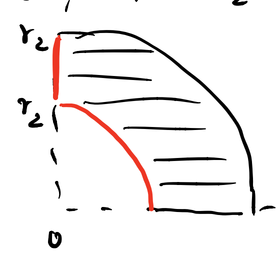


is simple

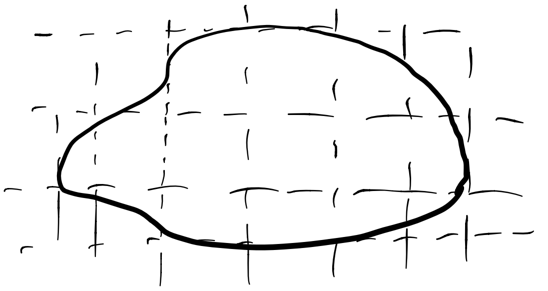
type (1): yes



type (2): yes



(iii')



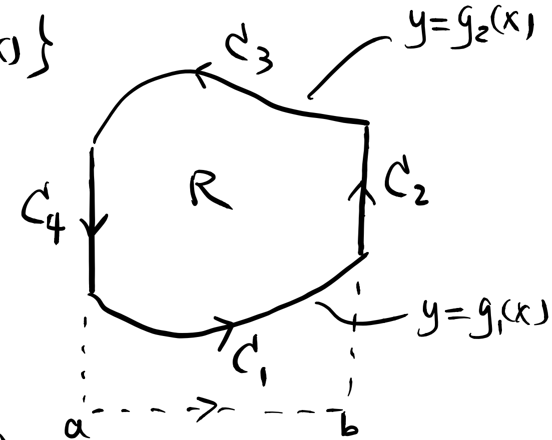
$\forall\, a \in \mathbb{R}$, $\#\{\partial R \cap \{x=a\}\} \leq 2$ and $\#\{\partial R \cap \{y=a\}\} \leq 2$ $\Big\} \Rightarrow$ simple.

(provided ∂R is piecewise smooth.)

Pf of Green Thm for simple region

By definition, R is of type (1) and can be written as

$$R = \{(x,y) :\ a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$$



Let denote the components of the boundary of R by C_1, C_2, C_3, C_4 as in the figure (Note that C_2 and/or C_4 could just be a point.)

Then $\partial R = C_1 + C_2 + C_3 + C_4$ as oriented curve

(using "+" instead of "$\cup$" to denote the orientation)

Now $C_1 = \{y = g_1(x)\}$ can be parametrized by

$$\vec{r}(t):\ x = t,\ y = g_1(t),\quad a \leq t \leq b$$

with the correct orientation

$$\therefore \int_{C_1} M\,dx = \int_a^b M(t, g_1(t))\,dt.$$

Similarly "$-C_3'$" can be parametrized by

$$\vec{r}(t): \quad x=t, \ y=g_2(t), \quad a \le t \le b$$

(with the correct orientation of $-C_3'$)

$$\therefore \quad \int_{-C_3'} M\,dx = \int_a^b M(t, g_2(t))\,dt$$

$$\Rightarrow \quad \int_{C_3'} M\,dx = -\int_{-C_3'} M\,dx = -\int_a^b M(t, g_2(t))\,dt$$

For $C_2' = \{x=b\}$, it can be parametrized by

$$\vec{r}(t): \quad x=b, \ y=t, \quad g_1(b) \le t \le g_2(b)$$

(with correct orientation)

$$\therefore \quad \int_{C_2'} M\,dx = 0 \qquad \left(\text{since } \frac{dx}{dt}=0\right)$$

Similarly $\int_{C_4'} M\,dx = -\int_{-C_4'} M\,dx = 0$.

Hence

$$\oint_{\partial R} M\,dx = \sum_{i=1}^{4} \int_{C_i'} M\,dx$$

$$= \int_a^b [M(t, g_1(t)) - M(t, g_2(t))]\,dt$$

(replacing the dummy variable $= \int_a^b [M(x, g_1(x)) - M(x, g_2(x))]\,dx$)

On the other hand, Fubini's Thm $\Rightarrow$

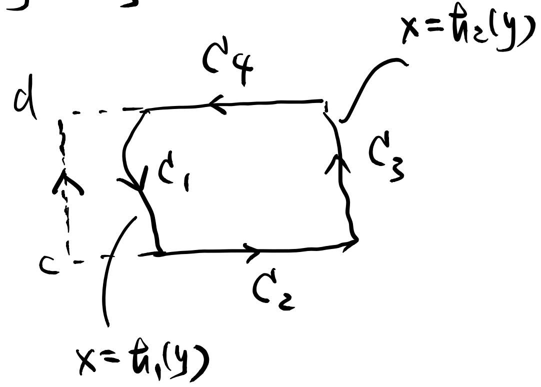
$$\iint_R -\frac{\partial M}{\partial y}\,dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} -\frac{\partial M}{\partial y}\,dy\right]dx$$

$$= \int_a^b -\left[M(x, g_2(x)) - M(x, g_1(x))\right]dx$$

$$\therefore \quad \iint_R -\frac{\partial M}{\partial y}\, dA = \oint_{\partial R} M\,dx .$$

Since R is also type (2), R can be written as

$$R = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$$



On can show similarly

$$\oint_{\partial R} N\,dy = -\int_c^d N(h_1(t), t)\,dt$$
$$+ 0$$
$$+ \int_c^d N(h_2(t), t)\,dt + 0$$
$$= \int_c^d [N(h_2(t), t) - N(h_1(t), t)]\,dt$$
$$= \int_c^d [N(h_2(y), y) - N(h_1(y), y)]\,dy$$
$$= \int_c^d \left[\int_{h_1(y)}^{h_2(y)} \frac{\partial N}{\partial x}\,dx\right] dy$$
$$= \iint_R \frac{\partial N}{\partial x}\,dA$$

All together $\quad \oint_{\partial R} M\,dx + N\,dy = \iint_R -\frac{\partial M}{\partial y}\,dA + \iint_R \frac{\partial N}{\partial x}\,dA$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA .$$

(Next time R = finite union of simple regions
(with certain condition)