Corollary (to Thm 9)

Let $\vec{F}$ be conservative and C^1

(n=3) If $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ (on $\Omega \subset \mathbb{R}^3$)

then
$$\begin{cases} \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \\[2ex] \dfrac{\partial N}{\partial z} = \dfrac{\partial L}{\partial y} \\[2ex] \dfrac{\partial L}{\partial z} = \dfrac{\partial M}{\partial z} \end{cases}$$

(n=2) If $\vec{F} = M\hat{i} + N\hat{j}$ (on $\Omega \subset \mathbb{R}^2$)

then
$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$$

Pf: $\vec{F}$ conservative $\xRightarrow{\text{Thm 9}}$ $\vec{F} = \vec{\nabla} f$ for some function f

i.e.
$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$
$$= M\hat{i} + N\hat{j} + L\hat{k} = \vec{F}$$

$\vec{F} \in C^1 \Rightarrow f \in C^2$. Hence Mixed derivatives theorem (Clairaut's Thm)

$$\begin{cases} \dfrac{\partial M}{\partial y} = \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial N}{\partial x} \\[2ex] \dfrac{\partial N}{\partial z} = \dfrac{\partial^2 f}{\partial z \partial y} = \dfrac{\partial^2 f}{\partial y \partial z} = \dfrac{\partial L}{\partial y} \\[2ex] \dfrac{\partial L}{\partial x} = \dfrac{\partial^2 f}{\partial x \partial z} = \dfrac{\partial^2 f}{\partial z \partial x} = \dfrac{\partial M}{\partial x} \end{cases}$$

\#

(Similarly for n=2)

eg 42: Show that $\vec{F}(x,y) = \hat{i} + x\hat{j}$ is not conservative on $\mathbb{R}^2$

Soln: $(\vec{F} \in C^\infty)$, $\begin{cases} M \equiv 1 \\ N = x \end{cases} \Rightarrow \dfrac{\partial M}{\partial y} = 0 \neq 1 = \dfrac{\partial N}{\partial x}$.

By Cor to Thm 9, $\vec{F}$ is not conservative. ※

Remark (Important)

For a C^1 vector field $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$

$$\boxed{\vec{F} \text{ conservative}} \;\underset{?}{\overset{\text{Cor to Thm 9}}{\Longrightarrow \atop \Longleftarrow}}\; \boxed{\text{M, N, L satisfy the system of PDE in Cor to Thm 9.}}$$

Answer: Not true in general, needs extra condition on the domain Ω.

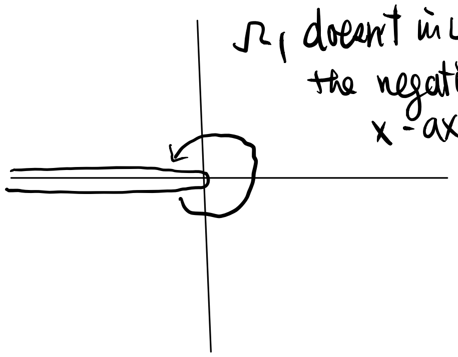
eg43 Consider the vector field

$$\vec{F} = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$$

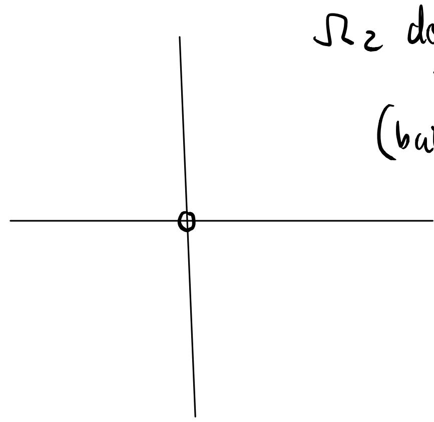
and the domains

$$\Omega_1 = \mathbb{R}^2 \setminus \{(x,0) \in \mathbb{R}^2 : x \leq 0\}$$
$$\Omega_2 = \mathbb{R}^2 \setminus \{(0,0)\}$$

Ω_1 doesn't include the negative x-axis

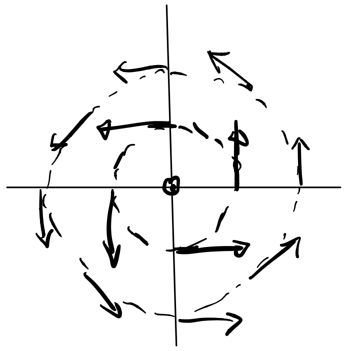


Ω_2 doesn't include the origin (but include negative x-axis)



In polar coordinates

$$\vec{F} = \frac{-\sin\theta}{r}\hat{i} + \frac{\cos\theta}{r}\hat{j}$$



$\Rightarrow$ $\vec{F}$ rotates around the origin anti-clockwisely

$|\vec{F}| = \frac{1}{r} \to 0$ as $r \to \infty$

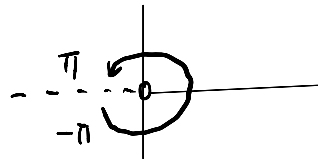
$|\vec{F}| = \frac{1}{r} \to +\infty$ as $r \to 0$ $\Rightarrow$ $\vec{F}$ cannot be extended to a C^1 vector field on $\mathbb{R}^2$.

Besides $(0,0)$, $\vec{F}$ is C^1 and hence $\vec{F}$ is C^1 on Ω_1 and also C^1 on Ω_2.

Questions : Is $\vec{F}$ conservative on Ω_1 ?
Is $\vec{F}$ conservative on Ω_2 ?

Soln: (1) For Ω_1, and (x,y) can be expressed in polar coordinates with

$$\begin{cases} r > 0 \\ -\pi < \theta < \pi \end{cases} \qquad ((r,\theta) \text{ are unique})$$



Define $f(x,y) = \theta$ "smooth" on Ω_1

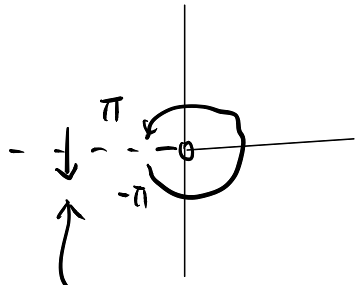
Then
$$\begin{cases} \frac{\partial f}{\partial x} = -\frac{\sin\theta}{r} \\ \frac{\partial f}{\partial y} = \frac{\cos\theta}{r} \end{cases} \qquad \left(\theta_x = -\frac{\sin\theta}{r}, \ \theta_y = \frac{\cos\theta}{r} \right)$$

$$\Rightarrow \quad \vec{F} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = \vec{\nabla} f .$$

$\Rightarrow$ $\vec{F}$ is conservative .

(2) For Ω_2, the function

$f(x,y)=\theta$ cannot be extended to a "smooth" function on (the whole) Ω_2



the function $f=\theta$ "jump" at the negative x-axis, $\Rightarrow f$ is <u>not</u> continuous. $\Rightarrow \vec{\nabla} f$ doesn't define on whole Ω_2

$\therefore f(x,y)=\theta$ doesn't work in the case of Ω_2

To show that $\vec{F}$ is <u>not</u> conservative, we consider a closed curve

$$C:\ \vec{r}(t)=\cos t\,\hat{i}+\sin t\,\hat{j},\quad t\in[-\pi,\pi]\quad \left(\begin{array}{c}\text{unit circle}\\ \text{in } \Omega_2\end{array}\right)$$

(not a curve in Ω_1)

Then

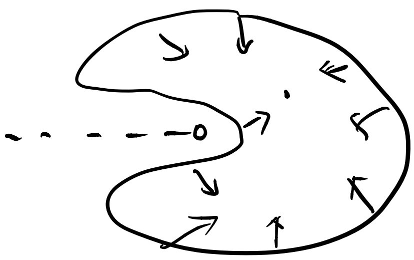
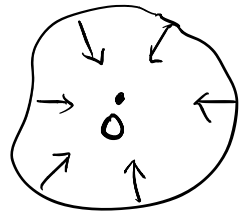
$$\oint_C \vec{F}\cdot d\vec{r}=\int_{-\pi}^{\pi}\left(-\frac{\sin\theta}{r}\hat{i}+\frac{\cos\theta}{r}\hat{j}\right)\cdot\vec{r}\,'(t)\,dt$$

$$=\int_{-\pi}^{\pi}(-\sin t\,\hat{i}+\cos t\,\hat{j})\cdot(-\sin t\,\hat{i}+\cos t\,\hat{j})\,dt$$

$$=\int_{-\pi}^{\pi}1\cdot dt = 2\pi \neq 0$$

By Thm 9, $\vec{F}$ is not conservative. $\#$

Summary:

Ω_1	Ω_2
$f(x,y)=\theta$ smooth function on Ω_1	$f(x,y)=\theta$ is <u>not</u> a smooth function on Ω_2 (θ cannot be well-defined on the whole Ω_2)
$C: x^2+y^2=1$ is <u>not</u> a curve in Ω_1 because $(-1,0)\in C$, $(-1,0)\notin\Omega_1$	$C: x^2+y^2=1$ is a closed curve in Ω_2
 Closed curves cannot circle around the origin $\Rightarrow$ closed curves can be deformed continuous (within Ω_1) to a point (in Ω_1)	 C enclosed the "hole" $\Rightarrow$ C cannot be deformed continuous (within Ω_2) to a point (in Ω_2)

Def 15 A subset $D \subseteq \mathbb{R}^n$, $n = 2$ or 3, is called <u>simply-connected</u> if every closed curve in D can be <u>contracted</u> to a point in D without ever leaving D.

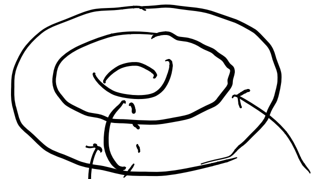
(contracted : deformed continuously)

<u>eg 44</u> : Ω_1 in eg 43 is simply-connected, but Ω_2 is <u>not</u> simply-connected.

<u>eg 45</u> : $S^2 \subset \mathbb{R}^3$ $\quad S^2 : x^2+y^2+z^2 = 1$

is simply-connected

<u>eg 46</u> :



torus $\mathbb{T}^2 \simeq S^1 \times S^1 \subset \mathbb{R}^3$ is <u>not</u> simply-connected

this closed curve cannot be contracted to a point on $\mathbb{T}^2$.

<u>Remark</u> : <u>Simply-connectedness</u> is a <u>global</u> condition to guarantee "eqts in Cor to Thm 9" $\Rightarrow$ "conservative".

<u>Thm 10</u> : Suppose $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, is <u>connected</u> and <u>simply-connected</u>. Let $\vec{F}$ be C^1 vector field on Ω. Then

$\vec{F}$ is conservative on Ω $\Leftrightarrow$ components of $\vec{F}$ satisfy the system of PDE in the Cor to the Thm. 9.

eg 47 : Let $\Omega \equiv \mathbb{R}^3$

(connected and simply-connected)

simply-connected, but not connected

Let $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$

$$= (y+e^z)\hat{i} + (x+1)\hat{j} + (1+xe^z)\hat{k}$$

Find the potential function f of $\vec{F}$, ie.

$$\vec{\nabla} f = \vec{F} .$$

Soln : This is, we want to solve

$$\frac{\partial f}{\partial x} = M, \quad \frac{\partial f}{\partial y} = N, \quad \frac{\partial f}{\partial z} = L .$$

Checking M, N, L satisfy the system of PDE in Cor to Thm 9 :

$$\begin{array}{ccc}
\frac{\partial M}{\partial x} = 0 & \frac{\partial M}{\partial y} = 1 & \frac{\partial M}{\partial z} = e^z \\
\frac{\partial N}{\partial x} = 1 & \frac{\partial N}{\partial y} = 0 & \frac{\partial N}{\partial z} = 0 \\
\frac{\partial L}{\partial x} = e^z & \frac{\partial L}{\partial y} = 0 & \frac{\partial L}{\partial z} = xe^z
\end{array}$$

Thm 10 $\Rightarrow$ existence of potential function f.

To find f explicitly :

$$\frac{\partial f}{\partial x} = y + e^z$$

$$\Rightarrow \quad f = \int (y+e^z)\,dx = x(y+e^z) + \text{"const in } x\text{"}$$

function indep. of x

$$= xy + xe^z + g(y,z) \quad \text{for some function } g(y,z)$$

$$\Rightarrow \quad x+1 = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(xy + xe^z + g(y,z)\right) = x + \frac{\partial g}{\partial y}$$

$$\Rightarrow \quad \frac{\partial g}{\partial y} = 1$$

$$\Rightarrow \quad g = y + \text{"const in } y\text{"}$$

$$= y + h(z) \quad \text{for some function } h(z)$$

$$\Rightarrow \quad f = xy + xe^z + y + h(z)$$

$$1 + xe^z = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(xy + xe^z + y + h(z)\right)$$

$$= xe^z + h'(z)$$

$$\Rightarrow \quad h'(z) = 1$$

$$\Rightarrow \quad h(z) = z + \text{const.}$$

Hence $f(x,y,z) = xy + xe^z + y + z + C$, where C is a constant,

is the required potential. . $\#$

(Note: This is equivalent to find f s.t.

the total differential $df = M\,dx + N\,dy + L\,dz$)

Remark: To prove Thm 10 in $\mathbb{R}^2$, we need the Green's Thm

(in $\mathbb{R}^3$, we need the Stokes' Thm)

Thm 11 (Green's Theorem)

Let $\Omega \subseteq \mathbb{R}^2$ be open, $\vec{F} = M\hat{i} + N\hat{j}$ be C^1 vector field on Ω;

C is a piecewise "smooth" simple closed anti-clockwisely oriented curve enclosing a region R which lies entirely in Ω.

Then

- Normal Form

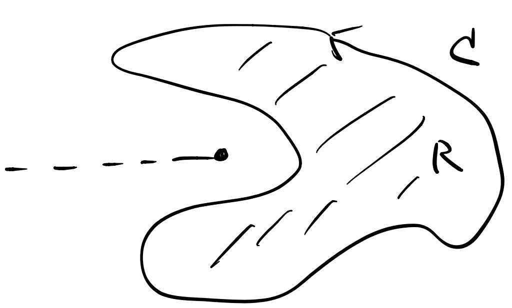
$$\oint_C \vec{F}\cdot\hat{n}\,ds = \boxed{\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy}$$

- Tangential Form

$$\oint \vec{F}\cdot\hat{T}\,ds = \boxed{\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$

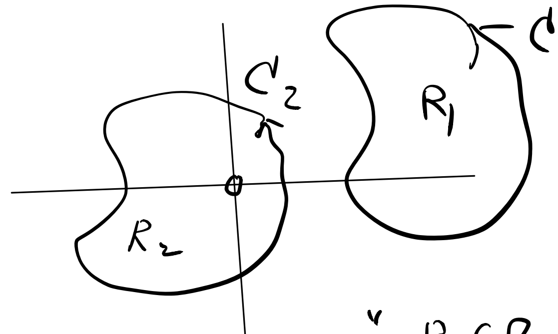
(Remark: The two forms are equivalent.)

<u>Note</u>: $\Omega_1 = \mathbb{R}^2 \setminus \{x \le 0\}$



Green's Thm applies on $R \subset \Omega_1$

$\Omega_2 = \mathbb{R}^2 \setminus \{(0,0)\}$



" $R_1 \subset \Omega_2$ "

Green's Thm applies

$R_2 \not\subset \Omega_2$

(since $(0,0) \notin \Omega_2$)

Green's Thm doesn't apply to C_2 & R_2!