eg 40 (Cont'd) Recall $\vec{F} = (x-y)\hat{i} + x\hat{j}$

$C: \ x^2+y^2=1$

Find the flow (anti-clockwisely) along C and flux across C.

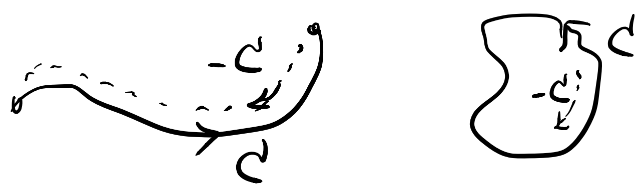
Soln: Let $\vec{r}(t) = \underset{(x(t))}{\cos t}\,\hat{i} + \underset{(y(t))}{\sin t}\,\hat{j}$, $0 \le t \le 2\pi$ (anti-clockwise!)

Then
$$\text{flow} = \oint_C \vec{F}\cdot\hat{T}\,ds = \oint_C \vec{F}\cdot d\vec{r}$$
$$= \int_0^{2\pi} [(\cos t - \sin t)\hat{i} + \cos t\,\hat{j}]\cdot[-\sin t\,\hat{i} + \cos t\,\hat{j}]\,dt$$
$$= \int_0^{2\pi} (-\cos t\sin t + 1)\,dt = 2\pi \quad \text{(check!)}$$

$$\text{flux} = \oint_C \vec{F}\cdot\hat{n}\,ds$$
$$= \oint_C M\,dy - N\,dx \quad \text{(with anti-clockwise parametrization)}$$
$$= \int_0^{2\pi} (\cos t - \sin t)\,d\sin t - \cos t\,d\cos t$$
$$= \int_0^{2\pi} [(\cos t - \sin t)\cos t + \cos t\sin t]\,dt$$
$$\cdots = \int_0^{2\pi} \cos^2 t\,dt = \pi \quad \text{(Check!)}$$

\#

Remark: If C is an oriented curve, denote by $-C$ the oriented curve with opposite orientation



- If f is a scalar function

$$\boxed{\int_C f\,ds = \int_{-C} f\,ds}$$

as "ds" is not oriented, just "length"

- If $\vec{F}$ is a vector field

flow $$\boxed{\int_C \vec{F}\cdot\hat{T}\,ds = -\int_{-C} \vec{F}\cdot\hat{T}\,ds}$$

this $\hat{T}$ is the $\hat{T}$ for $-C$

More precisely, we should write

$$\int_C \vec{F}\cdot\hat{T}_C\,ds = -\int_{-C} \vec{F}\cdot\hat{T}_{-C}\,ds$$

$$\& \quad \hat{T}_C = -\hat{T}_{-C}$$

- But for flux

$$\boxed{\oint_C \vec{F}\cdot\hat{n}\,ds = \oint_{-C} \vec{F}\cdot\hat{n}\,ds}$$

both are the same outward normal

Summary:

scalar f	$\int_C f\,ds$	independent of orientation	f, ds have no direction
vector $\vec{F}$ flow	$\int_C \vec{F}\cdot\hat{T}\,ds$	depends on orientation	$\hat{T}$ depends on orientation
flux	$\int_C \vec{F}\cdot\hat{n}\,ds$	independent of orientation	$\hat{n}$ = always outward

Conservative Vector Field

Def 14 Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, be open. A vector field $\vec{F}$ defined on Ω is said to be <u>conservative</u> if

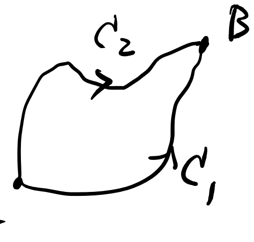
$$\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r} \text{ along an oriented curve } C \text{ in } \Omega$$

<u>depends</u> only on the <u>starting</u> point and <u>end</u> point of C.

<u>Note</u>: This is usually referred as "<u>path independent</u>".

i.e. If C_1, C_2 are <u>oriented</u> curves with <u>same</u> <u>starting</u> point A and <u>end</u> point B, then

$$\int_{C_1} \vec{F}\cdot\hat{T}\,ds = \int_{C_2} \vec{F}\cdot\hat{T}\,ds$$



(so the value only depends on the points A and B (& direction))

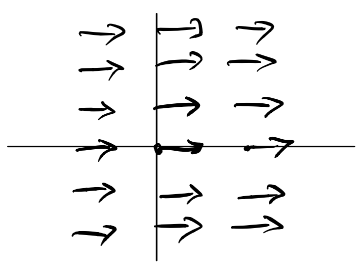
<u>Notation</u>: If $\vec{F}$ is conservative, we sometimes write

$$\int_A^B \vec{F}\cdot\hat{T}\,ds \text{ to denote the common value of}$$

$$\int_C \vec{F}\cdot\hat{T}\,ds \text{ along } \underline{\text{any}} \text{ oriented curve } C \text{ from } A \text{ to } B.$$

<u>eg 41</u>: $\vec{F} \equiv \hat{i}$ on $\mathbb{R}^2$

$C: \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, $a \le t \le b$



Then $\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_a^b x'(t)\,dt = x(b) - x(a)$

(x-coordinate at $\vec{r}(b)$ & at $\vec{r}(a)$)

$\therefore \int_C \vec{F}\cdot\hat{T}\,ds$ depends only on the starting & end points.

$\Rightarrow$ $\vec{F}$ is conservative.

(Note: $\vec{F} = \vec{\nabla} f$ where $f(x,y) = x$)

Thm 8 (Fundamental Theorem of Line Integral)

Let f be a C^1 function on an open set $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, and $\vec{F} = \vec{\nabla} f$ be gradient vector field of f. Then for any piecewise smooth oriented curve C on Ω with starting point A and end point B,

$$\boxed{\int_C \vec{F}\cdot\hat{T}\,ds = f(B) - f(A)}$$