Remarks: (1) "$ds = |\vec{r}'(t)|dt$" is usually refered as the arc-length element, where

$$\vec{r}'(t) = (x'(t), y'(t), z'(t))$$

and $$|\vec{r}'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

(2) Recall $x'(t) = \frac{dx}{dt}$, $y'(t) = \frac{dy}{dt}$, $z'(t) = \frac{dz}{dt}$ & $\vec{r}' = \frac{d\vec{r}}{dt}$.

Suppose the curve C is parametrized by a new parameter $\tilde{t}$

$$\underset{[a,b]}{\overset{t}{\cap}} \leftrightarrow \underset{[\tilde{a},\tilde{b}]}{\overset{\tilde{t}}{\cap}} \qquad \left(t \Leftrightarrow \tilde{t} \text{ is increasing} \right) \quad \frac{d\tilde{t}}{dt} > 0, \ \frac{dt}{d\tilde{t}} > 0$$

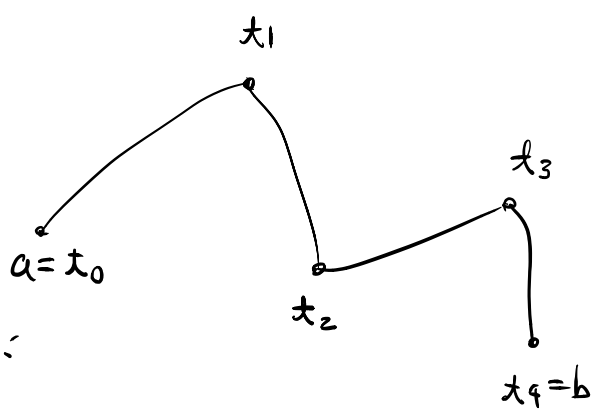
$$ds = |\vec{r}'(t)|dt = \left|\frac{d\vec{r}}{dt}(t)\right|dt$$

$$= \left|\frac{d\vec{r}}{d\tilde{t}}\frac{d\tilde{t}}{dt}\right|dt = \left|\frac{d\vec{r}}{d\tilde{t}}\right|\frac{d\tilde{t}}{dt}dt \quad \text{(by Chain rule)}$$

$$= \left|\frac{d\vec{r}}{d\tilde{t}}\right|d\tilde{t}.$$

$\therefore$ ds and hence $\int_C f(\vec{r})ds$ is <u>independent</u> of the parametrization of C.

(3) If $\vec{r}(t)$ is only piecewise differentiable, then the RHS of <u>Def 9'</u> becomes sum of each pieces:



If $[a,b] = [t_0, t_1] \cup \cdots \cup [t_{i-1}, t_i] \cup \cdots [t_{k-1}, t_k]$, where $t_0 = a$, $t_k = b$,

such that $\vec{r}\big|_{[t_{i-1}, t_i]}$ is differentiable, then $\boxed{\int_C f(\vec{r})\,ds = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} f(\vec{r})\,|\vec{r}'(t)|\,dt}$

eg32: $f(x,y,z) = x - 3y^2 + z$

C = line segment joining the origin and $(1,1,1)$

Find $\int_C f(x,y,z)\,ds$.

Soln: Parametrize C by

$$\vec{r}(t) = t(1,1,1) = (t,t,t), \quad t \in [0,1]$$

(i.e. $x(t)=t$, $y(t)=t$, $z(t)=t$)

$\Rightarrow \quad \vec{r}'(t) = (1,1,1), \quad \forall t \in [0,1]$

$\Rightarrow \quad |\vec{r}'(t)| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

Hence $\int_C f(x,y,z)\,ds = \int_0^1 f(t,t,t)\sqrt{3}\,dt$

$$= \int_0^1 (t - 3t^2 + t)\sqrt{3}\,dt = 0 \quad \text{(check!)}$$

#

eg33: Let C be curve in $\mathbb{R}^2$ (i.e. $z(t) \equiv 0$)

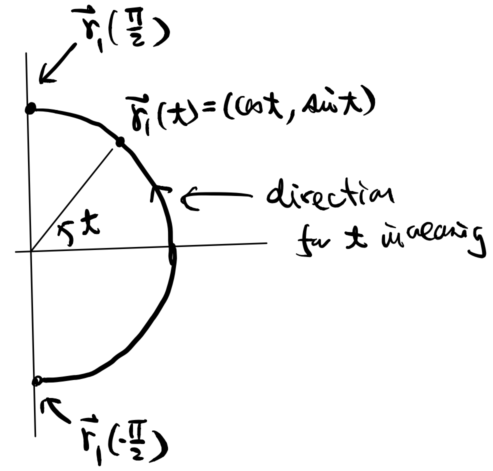
and its has 2 parametrizations:

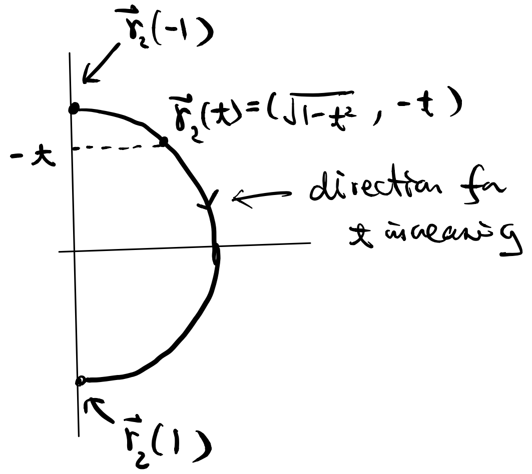
$$\vec{r}_1(t) = (\cos t, \sin t), \quad t \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

$$\vec{r}_2(t) = (\sqrt{1-t^2}, -t), \quad t \in [-1,1]$$

Suppose $f(x,y) = x$. Find $\int_C f(x,y)\,ds$.

(We simply omit the z-variable, as C is a plane curve and f is independent of z)





<u>Soln</u> : (i) $\vec{r}_1(t) = (\cos t, \sin t)$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

$$\vec{r}_1{}'(t) = (-\sin t, \cos t) \Rightarrow |\vec{r}\,'(t)| = 1$$

$$\Rightarrow \int_C f(x,y)\, ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\vec{r}_1(t))\, |\vec{r}\,'(t)|\, dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos t) \cdot 1\, dt = 2 \quad \text{(check!)}$$

(ii) $\vec{r}_2(t) = (\sqrt{1-t^2}, -t)$, $-1 \le t \le 1$

$$\int_C f(x,y)\, ds = \int_{-1}^{1} \sqrt{1-t^2} \left| \left(\frac{d}{dt}\sqrt{1-t^2}, \frac{d}{dt}(-t) \right) \right| dt$$

$$= \int_{-1}^{1} \sqrt{1-t^2} \left| \left(\frac{-t}{\sqrt{1-t^2}}, -1 \right) \right| dt$$

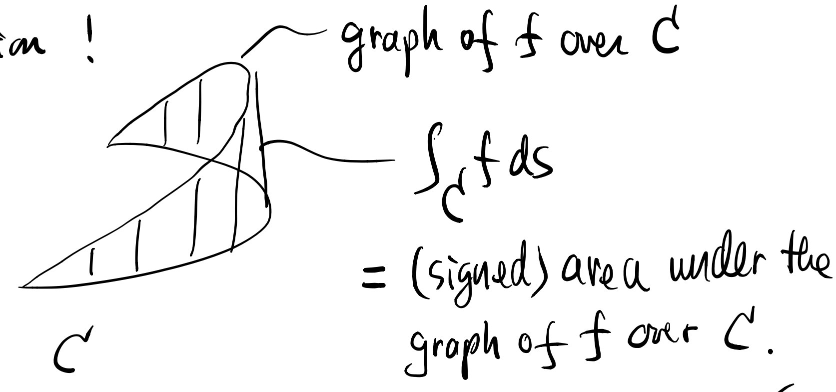
$$= \int_{-1}^{1} \sqrt{1-t^2} \sqrt{\left(\frac{-t}{\sqrt{1-t^2}}\right)^2 + (-1)^2}\, dt$$

$$= \int_{-1}^{1} dt = 2 \quad \text{(check!)}$$

or simply write

$$\left(\int_{-1}^{1} \sqrt{1-t^2} \cdot \sqrt{\left(\frac{d}{dt}\sqrt{1-t^2}\right)^2 + \left(\frac{d}{dt}(-t)\right)^2}\, dt \right)$$

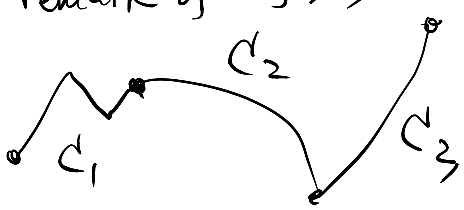
This verifies the fact that the line integral is independent of the parametrization !



$\#$

$\boxed{\begin{array}{l}\underline{\text{Prop 7}}: \text{If } C \text{ is a piecewise smooth curve made by joining } C_1, C_2, \cdots, C_n \text{ end to end, then} \\ \qquad \boxed{\displaystyle\int_C f\,ds = \sum_{i=1}^{n} \int_{C_i} f\,ds}\end{array}}$

(Pf: Clear from the remark of Def 8′)

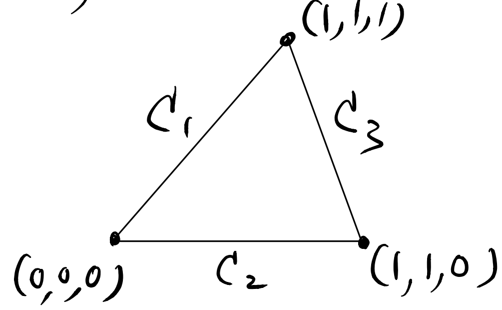


$C = C_1 \cup C_2 \cup C_3$

end point of C_{k-1} = initial (end) point of C_k.

__eg34__: Let $f(x,y,z) = x - 3y^2 + z$ (again)

C_1, C_2, C_3 are line segments as in the figure:



We already did $\int_{C_1} f\,ds = 0$ (eg 32).

One can similarly do $\int_{C_2 \cup C_3} f\,ds = \int_{C_2} f\,ds + \int_{C_3} f\,ds$

$$= -\frac{\sqrt{2}}{2} - \frac{3}{2}$$

(ex.)

The observation is $\int_{C_1} f\,ds = 0 \neq -\frac{\sqrt{2}}{2} - \frac{3}{2} = \int_{C_2 \cup C_3} f\,ds$

even C_1 & $C_2 \cup C_3$ have the same end points!

Conclusion:

Line integral of a function depends, not only on the end points, but also the path.