eg 27 (con't)

<u>Answer</u>: For $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2} = \dfrac{1}{\rho^2}$ (in spherical coordinates)

$B = \{0 < \rho \leq 1\}$

$$\lim_{\varepsilon \to 0} \iiint_{B \setminus B_\varepsilon} f(x,y,z)\, dV$$

$$= \lim_{\varepsilon \to 0} \int_0^{2\pi} \int_0^{\pi} \int_\varepsilon^1 \frac{1}{\rho^2} \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \lim_{\varepsilon \to 0} \int_0^{2\pi} \int_0^{\pi} \int_\varepsilon^1 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \lim_{\varepsilon \to 0} \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\phi\, d\phi\right)\left(\int_\varepsilon^1 d\rho\right)$$

$$= \lim_{\varepsilon \to 0} 4\pi(1-\varepsilon) = 4\pi \quad \text{(exists!)}$$

For $g(x,y,z) = \dfrac{1}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \dfrac{1}{\rho^3}$

$$\lim_{\varepsilon \to 0} \iiint_{B \setminus B_\varepsilon} g(x,y,z)\, dV$$

$$= \lim_{\varepsilon \to 0} \int_0^{2\pi} \int_0^{\pi} \int_\varepsilon^1 \frac{1}{\rho^3} \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \lim_{\varepsilon \to 0} \int_0^{2\pi} \int_0^{\pi} \int_\varepsilon^1 \frac{1}{\rho} \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \lim_{\varepsilon \to 0} \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\phi\, d\phi\right)\left(\int_\varepsilon^1 \frac{1}{\rho}\, d\rho\right)$$

$$= \lim_{\varepsilon \to 0} 4\pi \ln\frac{1}{\varepsilon} \quad \text{doesn't exist!}$$

<u>Terminology</u>: we said that $f = \dfrac{1}{\rho^2}$ is "integrable" in the sense of <u>improper integral</u>; $g = \dfrac{1}{\rho^3}$ is "not integrable" in the sense

of improper integral.

Question: determine all $\beta > 0$ such that

$$f = \frac{1}{\rho^\beta} \text{ is "integrable" in } B \subset \mathbb{R}^3.$$

Similar question in $\mathbb{R}^2$: determine all $\beta > 0$ such that

$$f = \frac{1}{r^\beta} \text{ is "integrabe" in } \underset{\{r \le 1\}}{D} \subset \mathbb{R}^2$$

(even in $\mathbb{R}^1$: $f = \frac{1}{|x|^\beta}$)

Application of Multiple integrals (Thomas' Calculus §15.6)

In applications, we often use the following:

In 2-dim: R is a region in $\mathbb{R}^2$ with density $\delta(x,y)$

- First moment about y-axis: $M_y = \iint_R x\,\delta(x,y)\,dA$
- First moment about x-axis: $M_x = \iint_R y\,\delta(x,y)\,dA$
- Mass: $M = \iint_R \delta(x,y)\,dA$
- Center of Mass (Centroid)

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$$

In 3-dim, D solid region in $\mathbb{R}^3$ with density $\delta(x,y,z)$

First moments:

- about yz-plane: $M_{yz} = \iiint_D x\,\delta(x,y,z)\,dV$
- about xz-plane: $M_{xz} = \iiint_D y\,\delta(x,y,z)\,dV$
- about xy-plane: $M_{xy} = \iiint_D z\,\delta(x,y,z)\,dV$
- Mass = $M = \iiint_D \delta(x,y,z)\,dV$
- Center of Mass (Centroid)

$$(\bar{x},\bar{y},\bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$$

In 2-dim, R region in $\mathbb{R}^2$ with density $\delta(x,y)$

Moment of inertia

- about x-axis: $I_x = \iint_R y^2\,\delta(x,y)\,dA$
- about y-axis: $I_y = \iint_R x^2\,\delta(x,y)\,dA$
- about line L: $I_L = \iint_R r(x,y)^2\,\delta(x,y)\,dA$

  where $r(x,y)$ = distance between (x,y) and L.

- about the origin: $I_O = \iint_R (x^2+y^2)\,\delta(x,y)\,dA$

In 3-dim, D = solid region in $\mathbb{R}^3$ with density $\delta(x,y,z)$

<u>Moments of Inertia</u>

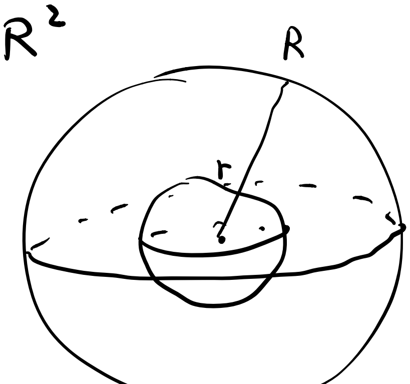
- around x-axis : $I_x = \iiint_D (y^2+z^2)\,\delta(x,y,z)\,dV$
- around y-axis : $I_y = \iiint_D (x^2+z^2)\,\delta(x,y,z)\,dV$
- around z-axis : $I_z = \iiint_D (x^2+y^2)\,\delta(x,y,z)\,dV$
- around line L : $I_L = \iiint_D r(x,y,z)^2\,\delta(x,y,z)\,dV$

where $r(x,y,z)$ = distance between (x,y,z) and L.

eg 28 : Consider $D: \; r^2 \le x^2+y^2+z^2 \le R^2$

with density $\delta(x,y,z) \equiv \delta$

constant density (i.e. uniform mass)



Express I_z in term of the mass

m = mass of D, r, and R.

<u>Soln</u> : $I_z \overset{\text{def}}{=} \iiint_D (x^2+y^2)\,\delta(x,y,z)\,dV$

$$= \delta \int_0^{2\pi}\int_0^{\pi}\int_r^{R} (\rho\sin\phi)^2\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \delta\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin^3\phi\,d\phi\right)\left(\int_r^R \rho^4\,d\rho\right)$$

$$= 2\pi \cdot \frac{4}{3}\cdot\frac{R^5-r^5}{5}\cdot\delta \qquad \text{(check)}$$

$$= \frac{8\pi}{15}(R^5-r^5)\,\delta$$

Mass $m = \iiint_D \delta(x,y,z)\,dV = \delta \iiint_D dV$

$$= \delta \cdot \frac{4\pi}{3}(R^3 - r^3)$$

$$\Rightarrow \quad \boxed{I_z = \frac{2m}{5}\,\frac{R^5 - r^5}{R^3 - r^3}}$$

<u>Observation</u>: Two limiting cases:

(i) $r \to 0$, i.e. the whole solid ball

$$\Rightarrow \quad \boxed{I_z = \frac{2m}{5}R^2}$$

(ii) $r \to R$ i.e. a (hollow) sphere made of infinitesimally thin sheet.

$$\Rightarrow \quad I_z = \lim_{r\to R} \frac{2m}{5}\,\frac{R^5 - r^5}{R^3 - r^3} = \frac{2m}{5} \cdot \frac{5R^4}{3R^2}$$

$$\therefore \quad \boxed{I_z = \frac{2m}{3}R^2}$$

(using $R^3 - r^3 = (R-r)(R^2 + Rr + r^2)$
$R^5 - r^5 = (R-r)(R^4 \cdots + r^4)$)

<u>Moment of inertia</u> of <u>hollow sphere</u>

> moment of inertia of the <u>solid ball</u>

(assuming the same uniform mass density)

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