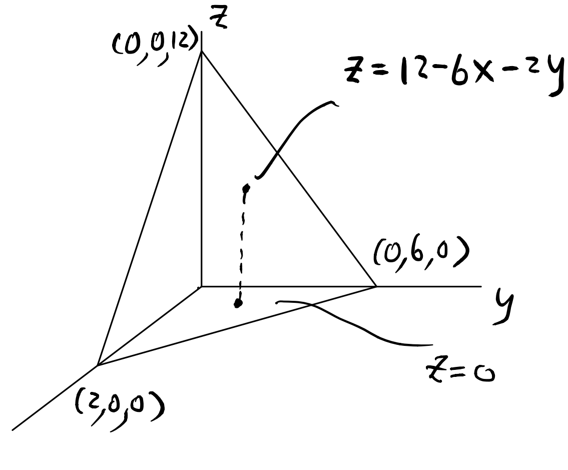
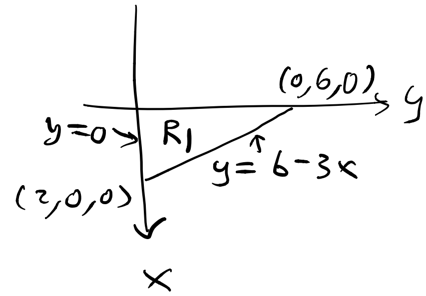
eg17 Volume of the bounded region D in the 1^{st} octant enclosed by the plane $6x+2y+z=12$



Soln: D is of special type

$$= \{ (x,y) \in R_1 : 0 \le z \le 12-6x-2y \}$$

$$= \{ 0 \le x \le 2,\ 0 \le y \le 6-3x,\ 0 \le z \le 12-6x-2y \}$$



$\Rightarrow$

$$\mathrm{Vol}(D) = \iiint_D 1 \cdot dV$$

$$= \int_0^2 \int_0^{6-3x} \int_0^{12-6x-2y} 1 \cdot dz\, dy\, dx$$

$$= \int_0^2 \int_0^{6-3x} (12-6x-2y)\, dy\, dx$$

$$= \int_0^2 (9x^2-36x+36)\, dx \quad \text{(check!)}$$

$$= 24 \quad \text{(Check!)} \qquad \text{(Compare eg22 later)}$$

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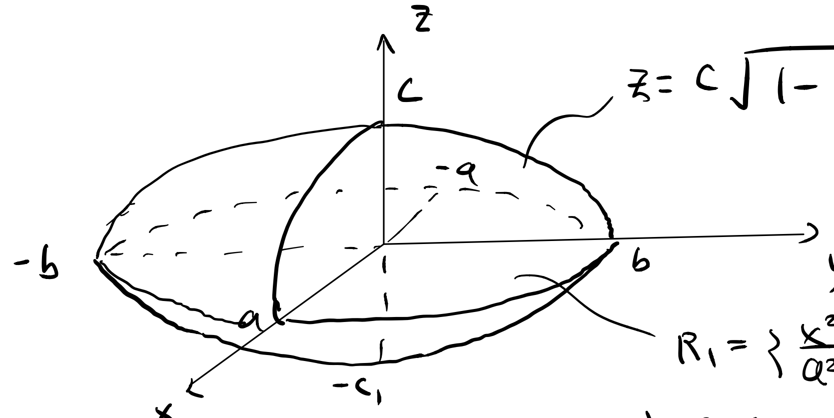
eg18 Volume of Ellipsoid

$$D = \left\{ (x,y,z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \right\} \quad (a,b,c>0)$$

In 1^{st} octant



$$z = c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

$$R_1 = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \right\},\ z=0$$

$$= \left\{ 0 \le x \le a,\ 0 \le y \le b\sqrt{1-\frac{x^2}{a^2}} \right\}$$

$\Rightarrow$ $\mathrm{Vol}(D) = 8 \times$ Vol of D in 1st octant (by symmetry)

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$$

$$= \cdots = \frac{4\pi abc}{3} \quad \text{(optimal exercise)}$$

In fact, we have, similarly

$$\mathrm{Vol}(D) = 8\int_0^c \int_0^{b\sqrt{1-\frac{z^2}{c^2}}} \int_0^{a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}} dx\,dy\,dz$$

and etc. (optimal exercise)

[Note: better way to do this is by change of variables formula (later in the course)]

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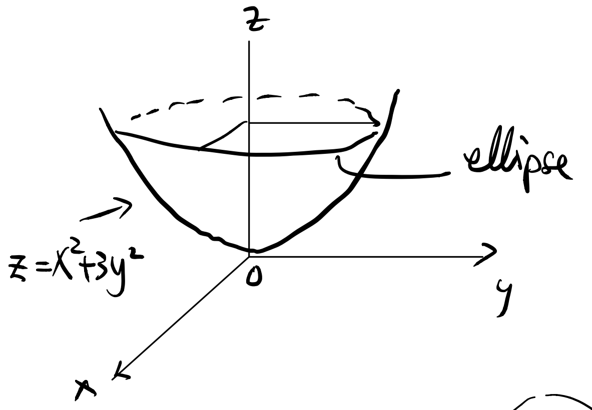
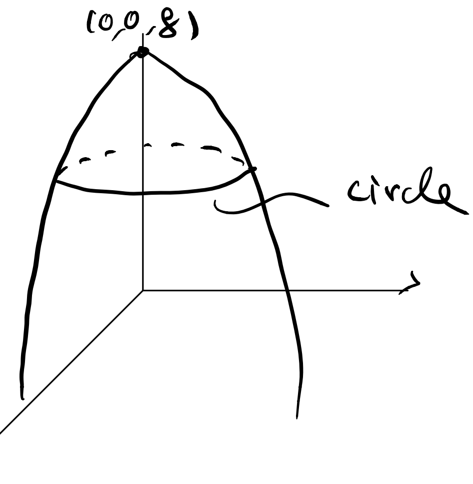
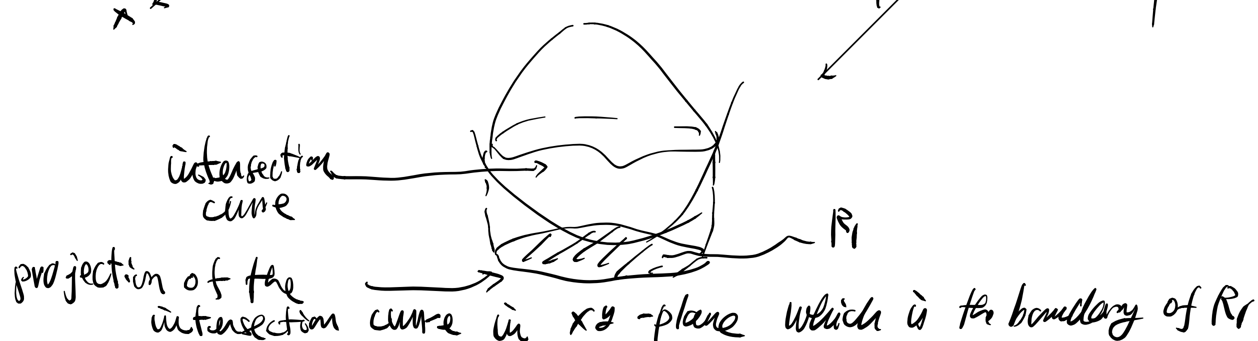
eg 19 Find the volume of D enclosed by

$$z = x^2+3y^2 \quad \text{and} \quad z = 8-x^2-y^2.$$





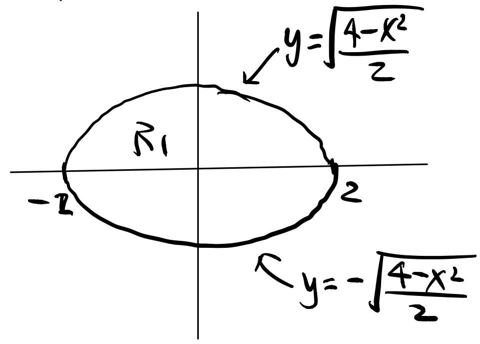


At the intersection of the two surfaces

$$x^2+3y^2 = z = 8-x^2-y^2$$

$$\Rightarrow \quad x^2+2y^2=4$$

$\Rightarrow$ the projection of the intersection curve in xy-plane is the ellipse $x^2+2y^2=4$



$$\Rightarrow D = \left\{ \begin{array}{l} (x,y)\in R_1=\{x^2+2y^2\le 4\}, \\ x^2+3y^2 \le z \le 8-x^2-y^2 \end{array} \right\}$$

$$= \left\{ \begin{array}{l} -2\le x\le 2, \ -\sqrt{\frac{4-x^2}{2}} \le y \le \sqrt{\frac{4-x^2}{2}}, \\ x^2+3y^2 \le z \le 8-x^2-y^2 \end{array} \right\}$$

$$\Rightarrow \mathrm{Vol}(D) = \int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8-2x^2-4y^2)\,dy\,dx$$

$$= \int_{-2}^{2} \frac{4\sqrt{2}}{3}(4-x^2)^{3/2}\,dx \quad \text{(check!)}$$

$$= 8\pi\sqrt{2} \quad \text{(check!)}$$

[For those interested in the intersection (space) curve (in parametric form):

$$x=2\cos t, \quad y=\sqrt{2}\sin t, \quad z=4+2\sin^2 t \quad (0\le t\le 2\pi)$$
]

eg20 Evaluate

$$\int_0^4 \int_0^1 \int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\, dx\, dy\, dz$$

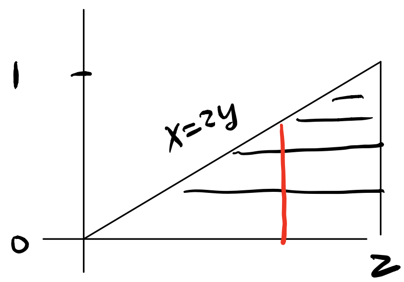
$$= \int_0^4 \frac{2}{\sqrt{z}} \Big[\underbrace{\int_0^1 \int_{2y}^2 \cos(x^2)\, dx\, dy}_{} \Big] dz$$

(z is a constant wrt xy integral)

this double integral doesn't depend on z.

$$= \left(\int_0^1 \int_{2y}^2 \cos(x^2)\, dx\, dy \right) \int_0^4 \frac{2}{\sqrt{z}}\, dz$$

think of this as double over the region



By Fubini's

$$= \left(\int_0^2 \int_0^{\frac{x}{2}} \cos x^2\, dy\, dx \right) \int_0^4 \frac{2}{\sqrt{z}}\, dz$$

$$= \left[\int_0^2 \left(\cos x^2 \int_0^{\frac{x}{2}} dy \right) dx \right] \int_0^4 \frac{2}{\sqrt{z}}\, dz$$

Then the integration can be easily evaluated !

$= 2\sin 4$ (Check!) ✗