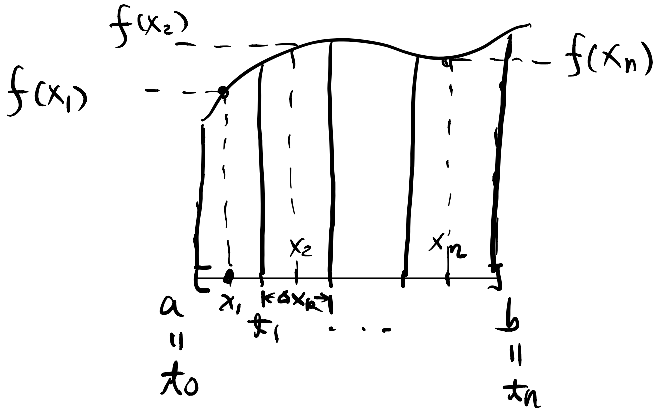
Double Integrals

Recall: In one-variable, "integral" is regarded as "limit" of "Riemann sum" (take MATH2060 for rigorous treatment)

$$\int_a^b f(x)dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k$$

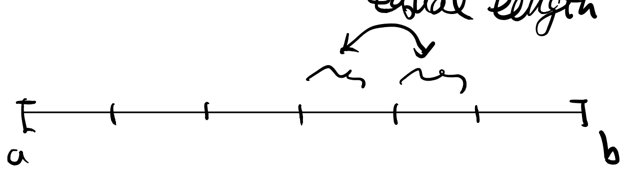
where
- f is a function on the interval $[a,b]$
- P is a partition $a = t_0 < t_1 < \cdots < t_n = b$
- $x_k \in [t_{k-1}, t_k]$ and $\Delta x_k = t_k - t_{k-1}$
- $\|P\| = \max_k |\Delta x_k|$



Remark: We usually use uniform partition P:

$$a = t_0 < t_1 = a + \frac{1}{n}(b-a) < t_2 = a + \frac{2}{n}(b-a) < \cdots$$

$$\cdots < t_k = a + \frac{k}{n}(b-a) < \cdots\cdots = t_n = b$$



In this case $\|P\| = \max_k |\Delta x_k| = \frac{b-a}{n} \to 0 \iff n \to \infty$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\cdot \Delta x_k \qquad \left(x_k \in [t_{k-1}, t_k] \right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\cdot \frac{b-a}{n}$$

<u>eg1</u>: Find $\int_0^1 x^2\,dx$ (i.e. $f(x)=x^2$ on $[0,1]$)

<u>Soln</u>: (1) One may choose $x_k = \frac{k-1}{n} \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$, where $\frac{k-1}{n} = t_{k-1}$, $\frac{k}{n} = t_k$

then

$$S_n = \sum_{k=1}^{n} x_k^2\,\Delta x_k$$

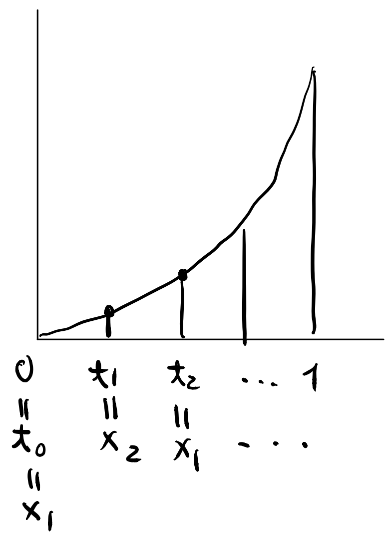
$$= \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)^2 \cdot \frac{1}{n}$$

(check!) $$= \frac{1}{n^3}\cdot\frac{(n-1)n(2n-1)}{6}$$

$$= \frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$

$$\longrightarrow \frac{2}{6} \quad \text{as } n\to\infty$$

$$\therefore \int_0^1 x^2\,dx = \frac{1}{3}$$



(2) Or we may choose $x_k = \frac{k}{n} \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$

(Will we get different answer?)

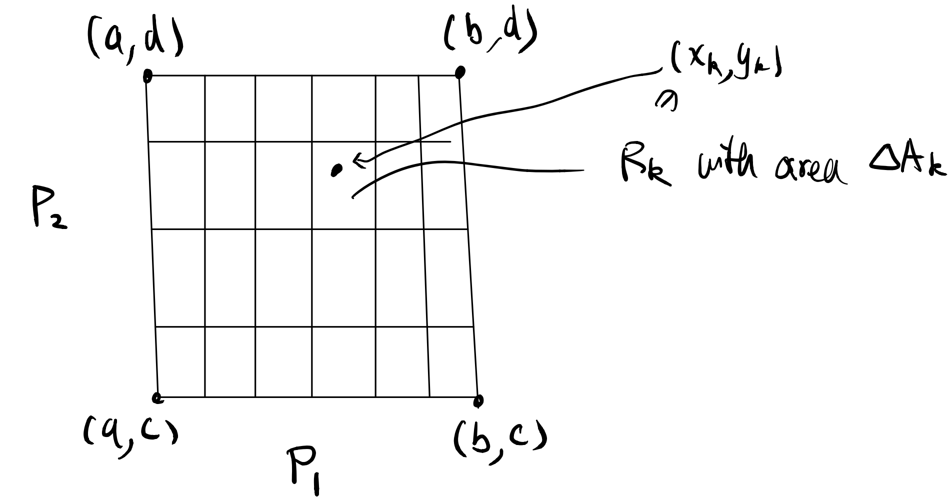
Then $S_n = \sum_{k=1}^{n} x_k^2\,\Delta x_k = \sum_{k=1}^{n}\left(\frac{k}{n}\right)^2\frac{1}{n} = \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$

$$\longrightarrow \frac{1}{3} \quad \text{as } n\to\infty$$

<u>Remark</u>: We can use any $x_k \in [t_{k-1}, t_k]$, and still get the same $\int_0^1 x^2\,dx = \frac{1}{3}$ #

This concept can be generalized to any dimension.

For 2-dim., let me first consider a function $f(x,y)$ defined on a rectangle $R = [a,b] \times [c,d] = \{(x,y) : a \le x \le b,\ c \le y \le d\}$



Then we can subdivide R into sub-rectangles by using partitions P_1 of $[a,b]$ and P_2 of $[c,d]$.

Denote $P = P_1 \times P_2$ (partition, subdivision, of R)

and $\|P\| = \max(\|P_1\|, \|P_2\|)$

Let the sub-rectangles be R_k, $k = 1, \cdots, N$ = number of subrectangles

with areas ΔA_k

Choose point $(x_k, y_k) \in R_k$, then consider "Riemann sum"

$$S(f; P) = \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k$$

Def 1 : The function f is said to be integrable over R

if $\lim_{\|P\| \to 0} S(f; P) = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k$

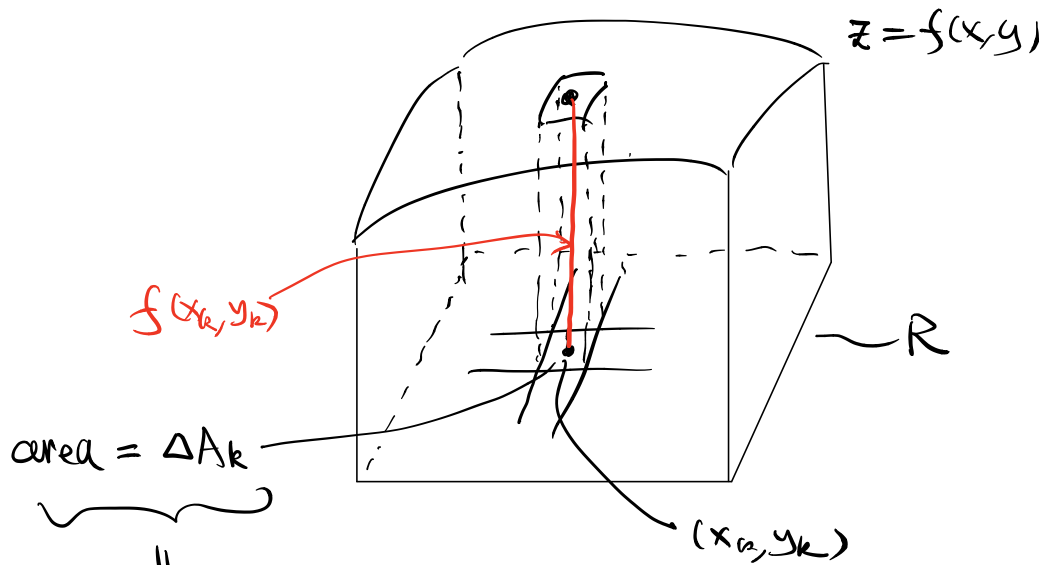
exists & independent of the choose of $(x_k, y_k) \in R_k$.

In this case, the limit is called the (double) integral

of f over R and is denoted by

$$\iint_R f(x,y)\,dA \quad \text{or} \quad \iint_R f(x,y)\,dx\,dy$$

Remark : Same as 1-variable, the double integral of f $(f \geq 0)$ over R can be interpreted as volume under the graph of f

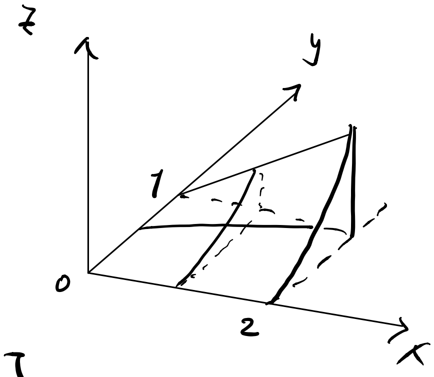


$\Downarrow$

$f(x_k, y_k)\Delta A_k \sim$ the volume under the graph of f over the sub-rectangle R_k

eg 2 $R = [0,2] \times [0,1]$, $f(x,y) = xy^2$

Find $\iint_R xy^2 \, dx\, dy$



Soln: Using the uniform partitions:

$P_1 = \{0, \frac{2}{n}, \frac{4}{n}, \cdots, 2\}$ of $[0,2]$

$P_2 = \{0, \frac{1}{n}, \frac{2}{n}, \cdots, 1\}$ of $[0,1]$

$\Rightarrow$ a particular subrectangle is

$$R_k = \left[\frac{2(i-1)}{n}, \frac{2i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right]$$

for some $i = 1, \cdots, n$; $j = 1, \cdots n$.

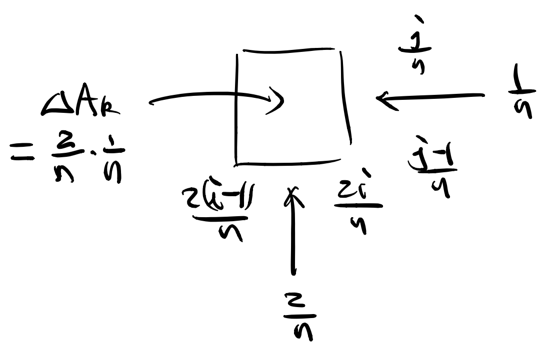
(So R_k should better denoted by R_{ij})

(Assume it is integrable)

One may choose the point $(x_k, y_k) = \left(\frac{2i}{n}, \frac{j}{n}\right) \in R_k$

and consider the Riemann sum

$$\sum_k f(x_k, y_k) \Delta A_k$$



$$= \sum_k (x_k y_k^2) \cdot \frac{2}{n^2}$$

$$= \sum_{i,j=1}^{n} \left[\frac{2i}{n} \cdot \left(\frac{j}{n}\right)^2\right] \cdot \frac{2}{n^2}$$

$$= \sum_{i,j=1}^{n} \frac{4}{n^5} i(j)^2 \quad = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{4}{n^5} i(j)^2$$

$$= \frac{4}{n^5} \sum_{i=1}^{n} \sum_{j=1}^{n} i j^2 = \frac{4}{5} \sum_{i=1}^{n} \left[i \sum_{j=1}^{n} j^2 \right]$$

$$= \frac{4}{n^5}\left(\sum_{i=1}^{n} i\right)\left(\sum_{j=1}^{n} j^2\right)$$

$$= \frac{4}{n^5} \cdot \frac{n(n+1)}{2} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$\rightarrow \frac{4 \cdot 2}{2 \cdot 6} = \frac{2}{3} \quad \text{as } n \to \infty$$

$$\therefore \iint_{[0,2]\times[0,1]} x y^2 \, dx\,dy = \frac{2}{3}$$

\#

Very tedious calculation!

Hence we need the following Theorem:

<u>Thm1</u> (Fubini's Theorem (1st form))

If $f(x,y)$ is <u>continuous</u> on $R = [a,b]\times[c,d]$,

then
$$\iint_R f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$
$$= \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

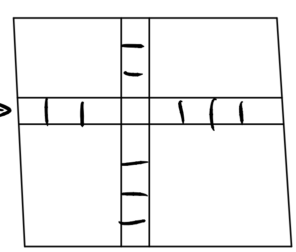
The last 2 integrals above are called <u>iterated integrals</u>.

(Pf: Omitted)

Ideas

sum horizontal 1st & taking limit

$\int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$



sum vertically first & taking limit

$\int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$

eg3 : Using Fubini to calculate $\iint_R xy^2\,dx\,dy$ where $R=[0,2]\times[0,1]$

Soln : By Fubini

$$\iint_R xy^2\,dA = \int_0^2\left[\int_0^1 xy^2\,dy\right]dx$$

$$= \int_0^2\left(x\int_0^1 y^2\,dy\right)dx$$

$$= \int_0^2\left(\frac{x}{3}\right)dx = \frac{2}{3}$$

$$\iint_R xy^2\,dA = \int_0^1\left(\int_0^2 xy^2\,dx\right)dy$$

$$= \int_0^1\left(y^2\int_0^2 x\,dx\right)dy$$

$$= \int_0^1 2y^2\,dy = \frac{2}{3}$$

Much easier than using Riemann sum! #

eg4 : Some times the "order" of the iterated integrals is important in practical calculations!

Find $\displaystyle\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA$.

Soln :

$$\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA = \int_0^\pi\left[\int_0^1 x\sin(xy)\,dx\right]dy$$

$$= \int_0^\pi\left[-\frac{\cos y}{y}+\frac{\sin y}{y^2}\right]dy \quad \text{(integration-by-parts)}$$

Not easy to integrate!

On the other hand, in different order

$$\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA = \int_0^1 \left(\int_0^{\pi} x\sin(xy)\,dy\right)dx$$

$$= \int_0^1 \Big[-\cos xy\Big]_{y=0}^{y=\pi} dx$$

$$= \int_0^1 (-\cos \pi x + 1)\,dx$$

$$= 1 \quad (\text{easy!})$$