MATH 2020 HW 7 sol

16.3) 10, 12, 21, 25, 27, 33b, 34

10)

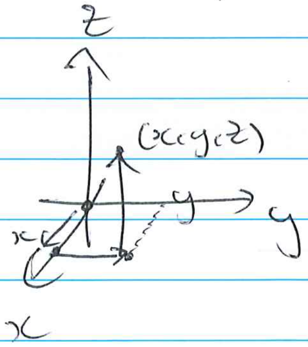
$$F = \begin{bmatrix} y\sin z \\ x\sin z \\ xy\cos z \end{bmatrix} = \begin{bmatrix} M \\ N \\ P \end{bmatrix}$$

By inspection, $f(x,y,z) = xy\sin z$ is a potential.
Alternatively, note that

$$\begin{cases} \partial_y M = \partial_x N = \sin z \\ \partial_z N = \partial_y P = x\cos z \\ \partial_x P = \partial_z M = y\cos z \end{cases}$$

so $f(x,y,z)$

$= \int_{(0,0,0)}^{(x,y,z)} (M\,dx + N\,dy + P\,dz)$ is well defined.



Consider the piecewise linear path with vertices $(0,0,0)$, $(x,0,0)$, $(x,y,0)$, (x,y,z).

$\int_{(0,0,0)}^{(x,0,0)} (M\,dx + N\,dy + P\,dz) = 0 \quad (\because M = N = P = 0)$

$\int_{(x,0,0)}^{(x,y,0)} (M\,dx + N\,dy + P\,dz)$

$= \int_{(x,0,0)}^{(x,y,0)} P\,dz \quad (\because M = N = 0)$

$= 0 \quad (\because$ path has no change in $z)$

$\int_{(x,y,0)}^{(x,y,z)} (M\,dx + N\,dy + P\,dz)$

$= \int_{(x,y,0)}^{(x,y,z)} P\,dz \quad (\because$ path has no change in x or $y)$

$= xy \int_0^z \cos t\,dt$

$= xy\sin z$

$\therefore f(x,y,z) = xy\sin z$.

12).

$$F = \begin{bmatrix} \frac{y}{1+x^2y^2} \\ \frac{x}{1+x^2y^2} + \frac{z}{\sqrt{1-y^2z^2}} \\ \frac{y}{\sqrt{1-y^2z^2}} + \frac{1}{z} \end{bmatrix}$$

By inspection

$$f(x,y,z) = \tan^{-1}(xy) + \sin^{-1}(yz) + \log|z|.$$

21)

let $F = \begin{bmatrix} \frac{1}{y} \\ \frac{1}{z} - \frac{x}{y^2} \\ -\frac{y}{z^2} \end{bmatrix}$ and

$$f(x,y,z) = \frac{x}{y} + \frac{y}{z}.$$

Then $F = \nabla f$.

hence $\int_{(1,1,1)}^{(2,2,2)} \left[\frac{1}{y}\,dx + \left(\frac{1}{z} - \frac{x}{y^2}\right)dy - \frac{y}{z^2}\,dz\right]$

$= f(2,2,2) - f(1,1,1)$

$= (1+1) - (1+1)$

$= 0.$

25) Observe $(z^2, 2y, 2xz) = \nabla(xz^2 + y^2)$,

so $\int_A^B (z^2\,dx + 2y\,dy + 2xz\,dz) = (xz^2+y^2)|_A - (xz^2+y^2)|_B$

is independent of path.

Alternatively, since the domain is $\mathbb{R}^2$, which is simply-connected, it suffices to apply component test.

$$\begin{cases} \partial_y z^2 = \partial_x 2y = 0 \\ \partial_z 2y = \partial_y 2xz = 0 \\ \partial_x 2xz = \partial_z z^2 = 2z \end{cases}$$

The result then follows.

27) $F = \begin{bmatrix} 2x/y \\ (1-x^2)/y^2 \end{bmatrix}$

let $f(x,y,z) = -\frac{1}{y} + \frac{x^2}{y}$

Then f is a potential of F.

33b). Since the domain is $\mathbb{R}^3$, which is simply-connected, it suffices to apply component test

F is a gradient

$$\Leftrightarrow \begin{cases} \partial_y (y^2 + 2czx) = \partial_x [y(bx + cz)] \\ \partial_z [y(bx + cz)] = \partial_y (y^2 + cx^2) \\ \partial_x (y^2 + cx^2) = \partial_z (y^2 + 2czx) \end{cases}$$

$$\Leftrightarrow \begin{cases} 2y = by \\ cy = 2y \\ 2cx = 2cx \end{cases}$$

$$\Leftrightarrow \quad b = 2, \quad c = 2$$

34) It suffices to show $\partial_x g = M$,

where $F = (M, N, P)$.

By path independence,

$$g(x+h, y, z) = \int_{\gamma_1} F \cdot dr + \int_{\gamma_2} F \cdot dr .$$

where γ_1 is a path from $(0,0,0)$ to (x,y,z),

γ_2 is $t \mapsto (x+t, y, z)$.

$$\partial_x g(x,y,z) = \frac{d}{dh}\Big|_{h=0} g(x+h, y, z)$$

$$= \frac{d}{dh}\Big|_{h=0} \int_0^h M(x+t, y, z)\, dt$$

($\because$ γ_2 has no change in y, z)

$$= M(x,y,z)$$

(by fund thm of cal)