MATH 2020 Advanced Calculus II

Solutions to HW6

Due date: 18 Oct

Practice exercises.

15. Cartesian coordinates:

$$\begin{aligned}
\text{Volume} &= \int_0^1 \int_x^{2-x} (x^2+y^2)dydx \\
&= \int_0^1 \left\{ x^2[(2-x)-x] + \frac{1}{3}[(2-x)^3 - x^3] \right\} dx \\
&= \int_0^1 \left(2x^2 - 2x^3 + \frac{8}{3} - 4x + 2x^2 - \frac{2x^3}{3} \right) dx \\
&= \frac{2}{3} - \frac{2}{4} + \frac{8}{3} - \frac{4}{2} + \frac{2}{3} - \frac{2}{3}\cdot\frac{1}{4} \\
&= \frac{4}{3}
\end{aligned}$$

Cylindrical coordinates:

$$\begin{aligned}
\text{Volume} &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{2}{\cos\theta+\sin\theta}} r^2 \cdot r dr d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{4}\left(\frac{2}{\cos\theta+\sin\theta}\right)^4 d\theta \\
&= 4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1+\tan^2\theta}{(1+\tan\theta)^4} \cdot \sec^2\theta d\theta \\
&= 4\int_1^{\infty} \frac{1+t^2}{(1+t)^4} dt \\
&= 4\int_1^{\infty} \left[\frac{1}{(1+t)^2} - \frac{2}{(1+t)^3} + \frac{2}{(1+t)^4}\right] dt \\
&= 4\left[\frac{1}{2} - \frac{2}{2\cdot 2^2} + \frac{2}{3\cdot 2^3}\right] \\
&= \frac{4}{3}
\end{aligned}$$

19.

$$\begin{aligned}
& \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2dydx}{(1+x^2+y^2)^2} \\
=\; & \int_0^{2\pi} \int_0^1 \frac{2rdrd\theta}{(1+r^2)^2} \\
=\; & 2\pi \left[\frac{-1}{1+r^2}\right]_0^1 \\
=\; & 2\pi\left(1 - \frac{1}{2}\right) \\
=\; & \pi
\end{aligned}$$

29.

$$\iiint_S dV = \int_0^1\int_0^3\int_0^1 dzdydx$$
$$= 1 \times 3 \times 1$$
$$= 3$$

$$\begin{aligned}\iiint_S 30xz\sqrt{x^2+y}dV &= \int_0^1\int_0^3\int_0^1 30xz\sqrt{x^2+y}dzdydx \\ &= \frac{30}{2}\int_0^1\int_0^3 x\sqrt{x^2+y}dydx \\ &= 15\int_0^1 \left[x\cdot\frac{2\sqrt{x^2+y}^3}{3}\right]_0^3 dx \\ &= 10\int_0^1 x\left(\sqrt{x^2+3}^3 - x^3\right)dx \\ &= 10\left[\frac{\sqrt{x^2+3}^5}{5} - \frac{x^5}{5}\right]_0^1 \\ &= 2(31-9\sqrt{3})\end{aligned}$$

So the average of f over S is

$$\frac{\iiint_S 30xz\sqrt{x^2+y}dV}{\iiint_S dV} = \frac{2(31-9\sqrt{3})}{3}.$$

35.

$$\int_0^{\frac{\pi}{2}}\int_1^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} r^3(\sin\theta\cos\theta)z^2dzdrd\theta$$
$$= \int_0^{\frac{\pi}{2}}\int_1^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} (r\sin\theta)(r\cos\theta)z^2dz(rdrd\theta)$$
$$= \int_0^1\int_{\sqrt{1-x^2}}^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}} yxz^2dzdydx + \int_1^{\sqrt{3}}\int_0^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}} yxz^2dzdydx$$

Remark. The above integral can be written as one single (triple) integral:

$$\int_0^{\sqrt{3}}\int_{\sqrt{\max(0,1-x^2)}}^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}} yxz^2dzdydx.$$

But the price we have to pay is that the lower limit for y which is a function of x is no longer smooth (only continuous). This expression, despite being in simpler form, does not simplify the computation.

36a.
- $-\sqrt{4-x^2-y^2} \leqslant z \leqslant \sqrt{4-x^2-y^2}$ represents the sphere of radius 2 and center the origin.
- Notice $y = \sqrt{2x-x^2} \iff (x-1)^2+y^2 = 1$, so $0 \leqslant y \leqslant \sqrt{2x-x^2}$ represents the right cylinder over the upper half disk of radius 1 and center $(1,0,0)$.

- The segment $\{(t,0,0) \mid 0 \leqslant t \leqslant 2\}$ is a diameter of this upper half disk.

It follows that the solid is bounded by $x^2+y^2+z^2=4$ (two components), $y=0$ and $(x-1)^2+y^2=1$.

36b.

$$\int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r dz dr d\theta$$

53.

$$\begin{cases} u = x - y \\ v = y \end{cases} \Longleftrightarrow \begin{cases} x = u + v \\ y = v \end{cases}$$

So

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1.$$

Now the region $R = \{0 \leqslant y \leqslant x\}$ is transformed into $R' = \{u, v \geqslant 0\}$. Hence

$$\begin{aligned} &\int_0^\infty \int_0^x e^{-sx} f(x-y,y) dy dx \\ = &\iint_{R'} e^{-s(u+v)} f(u,v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du dv \\ = &\int_0^\infty \int_0^\infty e^{-s(u+v)} f(u,v) du dv. \end{aligned}$$

Additional and advanced exercises.

12a.
- $x = \sqrt{a^2-y^2} \Longleftrightarrow r = a$ (and $x \geqslant 0$)
- $x = y \cot\beta \Longleftrightarrow \theta = \beta$
- $y = 0 \Longleftrightarrow \theta = 0$ (given $x \geqslant 0$)

It follows that the region over which the integral is defined is the circular sector with angle β and radius a lying in the first quadrant such that one of its sides lies in the positive x-axis. Using polar coordinates, we have

$$\begin{aligned} &\int_0^{a\sin\beta} \int_{y\cot\beta}^{\sqrt{a^2-y^2}} \ln(x^2+y^2) dx dy \\ = &\int_0^\beta \int_0^a \ln(r^2) r dr d\theta \\ = &\ 2\beta \left\{ \left[\ln(r) \cdot \frac{r^2}{2}\right]_0^a - \int_0^a \frac{r}{2} dr \right\} \\ = &\ 2\beta \left[\ln(a) \cdot \frac{a^2}{2} - \frac{a^2}{4}\right] \qquad \left(\because \lim_{t\to 0^+} \ln(t) t^2 = 0\right) \\ = &\ a^2\beta \left(\ln(a) - \frac{1}{2}\right). \end{aligned}$$

12b.

$$\int_0^{a\cos\beta} \int_0^{x\tan\beta} \ln(x^2+y^2) dy dx + \int_{a\cos\beta}^a \int_0^{\sqrt{a^2-x^2}} \ln(x^2+y^2) dy dx$$

Remark. The integral can also be written as

$$\int_0^a \int_0^{\min(x\tan\beta,\sqrt{a^2-x^2})} \ln(x^2+y^2)dydx.$$

Again, like Q35 above, this does not simplify the computation.

27.

$$\begin{aligned}
\iiint_D z(r^2+z^2)^{-\frac{5}{2}}dV &= \int_0^{2\pi}\int_0^1\int_1^{\infty} z(r^2+z^2)^{-\frac{5}{2}} r dz dr d\theta \\
&= 2\pi \int_0^1 \left[(r^2+z^2)^{-\frac{3}{2}}\left(-\frac{2}{3}\right)\left(\frac{1}{2}\right)\right]_1^{\infty} r dr \\
&= \frac{2\pi}{3}\int_0^1 r(r^2+1)^{-\frac{3}{2}}dr \\
&= \frac{2\pi}{3}\left[(r^2+1)^{-\frac{1}{2}}\left(-\frac{2}{1}\right)\left(\frac{1}{2}\right)\right]_0^1 \\
&= \frac{2\pi}{3}\left(1-\frac{1}{\sqrt{2}}\right) \\
&= \frac{(2-\sqrt{2})\pi}{3}
\end{aligned}$$