MATH2020A Homework 4

(15.7)

9.

$$\begin{aligned}
\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} (r^2 \cos^2\theta + z^2) r d\theta dr dz &= \int_0^1 \int_0^{\sqrt{z}} (r^2\pi + 2\pi z^2) r dr dz \\
&= \int_0^1 (\frac{\pi z^2}{4} + \pi z^3) dz \\
&= \frac{\pi}{12} + \frac{\pi}{4} \\
&= \frac{\pi}{3}
\end{aligned}$$

15.

$$\iiint_D f(r,\theta,z) dz r dr d\theta = \int_0^{\pi} \int_0^{2\sin\theta} \int_0^{4 - r\sin\theta} f(r,\theta,z) dz r dr d\theta$$

20.

$$\iiint_D f(r,\theta,z) dz r dr d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{1}{\sin\theta}} \int_0^{2 - r\cos\theta} f(r,\theta,z) dz r dr d\theta$$

29.

$$\begin{aligned}
\int_0^1\int_0^{\pi}\int_0^{\frac{\pi}{4}} 12\rho\sin^3\phi d\phi d\theta d\rho &= \left(\int_0^{\frac{\pi}{4}}\sin^3\phi d\phi\right)\left(\int_0^{\pi}d\theta\right)\left(\int_0^1 12\rho d\rho\right)\\
&= \left(\int_0^{\frac{\pi}{4}}\sin\phi(1-\cos^2\phi)d\phi\right)\times\pi\times 6\\
&= 6\pi\left[-\cos\phi+\frac{\cos^3\phi}{3}\right]_0^{\frac{\pi}{4}}\\
&= \frac{8-5\sqrt{2}}{2}\pi
\end{aligned}$$

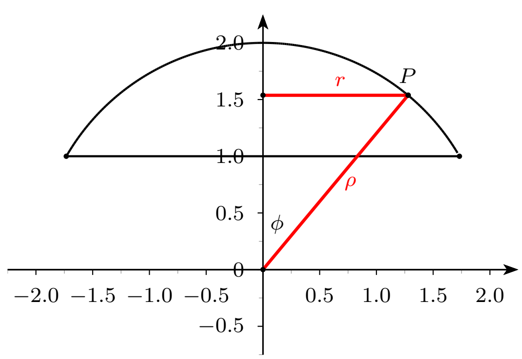
35.

$$\begin{aligned}
V &= \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\phi}\rho^2\sin\phi d\rho d\phi d\theta\\
&= \int_0^{2\pi}\int_0^{\pi}\frac{(1-\cos\phi)^2\sin\phi}{2}d\phi d\theta\\
&= \int_0^{2\pi}\left[\frac{(1-\cos\phi)^3}{6}\right]_0^{\pi}d\theta\\
&= 2\pi\times\frac{4}{3}\\
&= \frac{8}{3}\pi
\end{aligned}$$

41. By definition we have

$$D=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2\le 4, z\ge 1\}$$

Hence, range of θ is $[0,2\pi]$, range of ϕ is $[0,\frac{\pi}{3}]$, range of ρ is $[\frac{1}{\cos\phi},2]$



(a)

$$V(D) = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_{\frac{1}{\cos\theta}}^{2} \rho^2 \sin\phi d\rho d\phi d\theta$$

(b)

$$V(D) = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} dzrdrd\theta$$

(c)

$$V(D) = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} dzdydx$$

(d)

$$\begin{aligned} V(D) &= \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} dzrdrd\theta \\ &= \int_0^{2\pi} \int_0^{\sqrt{3}} (\sqrt{4-r^2} - 1) rdrd\theta \\ &= \int_0^{2\pi} \left[-\frac{1}{3}(4-r^2)^{\frac{3}{2}} - \frac{1}{2}r^2 \right]_0^{\sqrt{3}} d\theta \\ &= \frac{5\pi}{3} \end{aligned}$$

45.

$$\begin{aligned} V &= \int_{-\frac{\pi}{2}}^{0} \int_0^{3\cos\theta} \int_0^{-r\sin\theta} dzrdrd\theta \\ &= \int_{-\frac{\pi}{2}}^{0} \int_0^{3\cos\theta} -r\sin\theta rdrd\theta \\ &= \int_{-\frac{\pi}{2}}^{0} -\frac{1}{3}(3\cos\theta)^3 \sin\theta d\theta \\ &= 9\left[\frac{\cos^4\theta}{4}\right]_{-\frac{\pi}{2}}^{0} \\ &= \frac{9}{4} \end{aligned}$$

54.

$$
\begin{aligned}
V &= \int_0^{2\pi}\int_0^1\int_{r^2}^{r^2+1} dzrdrd\theta \\
&= \int_0^{2\pi}\int_0^1 rdrd\theta \\
&= \int_0^{2\pi}\frac{1}{2}d\theta \\
&= \pi
\end{aligned}
$$

62. Put $x^2+y^2=z$ in $x^2+y^2+z^2=2$ and we will get $z+z^2=2$. So the solution is $z=1$. So the range for this region is

$$\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\le 1, x^2+y^2\le z\le\sqrt{2-x^2-y^2}\}$$

Choose cylindrical Coordinates,

$$
\begin{aligned}
V &= \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} dzrdrd\theta \\
&= \int_0^{2\pi}\int_0^1 (r\sqrt{2-r^2}-r^3)drd\theta \\
&= \int_0^{2\pi}\frac{2\sqrt{2}-1}{3}-\frac{1}{4}d\theta \\
&= \frac{8\sqrt{2}-7}{6}\pi
\end{aligned}
$$

65.

$$
\begin{aligned}
\iiint\limits_R f(\rho,\phi,\theta)\rho^2\sin\phi d\rho d\phi d\theta &= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^3\sin\phi d\rho d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\frac{1}{4}\sin\phi d\phi d\theta \\
&= \int_0^{2\pi}\frac{1}{2}d\theta \\
&= \pi
\end{aligned}
$$

The volume of unit solid ball is $V=\frac{4}{3}\pi$. So the average is

$$\text{Average value}=\frac{\pi}{\frac{4}{3}\pi}=\frac{3}{4}$$