Math 1010 Week 6

Implicit Differentiation, Higher Order Derivatives

6.1 Implicit Differentiation

Example 6.1. *For $x > 0$,*

$$\frac{d}{dx}\ \ln x = \frac{1}{x}\ .$$

Proof. Consider the equation:

$$e^{\ln x} = x$$

Differentiating both sides with respect to x, and applying the Chain Rule, we have:

$$\frac{d}{dx}\ e^{\ln x} = \frac{d}{dx}\ x$$
$$\underbrace{e^{\ln x}}_{=x} \frac{d}{dx}\ \ln x = 1$$

Hence, $\dfrac{d}{dx}\ \ln x = \dfrac{1}{x}$. □

Example 6.2. *Find $\dfrac{d}{dx}\,(x^x)$, where $x > 0$.*

For any $x > 0$, we have $x = e^{\ln x}$. Hence,

$$x^x = \left(e^{\ln x}\right)^x = e^{x \ln x}.$$

So,

$$\begin{aligned}
\frac{d}{dx}\left(x^x\right) &= \frac{d}{dx}e^{x\ln x} \\
&= e^{x\ln x}\frac{d}{dx}(x\ln x) \quad (\textit{ by the Chain Rule.}) \\
&= e^{x\ln x}\left(x\cdot\frac{1}{x}+\ln x\right) \quad (\textit{ by the Product Rule.}) \\
&= e^{x\ln x}(1+\ln x) \quad (\textit{ since } x>0.) \\
&= (1+\ln x)x^x.
\end{aligned}$$

Exercise 6.3. *Consider the curve $C : y^4 - y\cos(x) - x^4 = 0$.*

1. *Find $\dfrac{dy}{dx}$. Express your answer in terms of x, y only.*
2. *Let $P = \left(\dfrac{\pi}{2}\ , -\dfrac{\pi}{2}\right)$.*
   - *Verify that the point P lies on the curve C.*
   - *Find the equation of the tangent line to the curve C at the point P.*

Solution. *First, we differentiate both sides of the equation $y^4 - y\cos(x) - x^4 = 0$ with respect to x:*

$$\frac{d}{dx}(y^4 - y\cos(x) - x^4) = \frac{d}{dx}0 \tag{6.1}$$

By the chain rule, we have:

$$\frac{d}{dx}y^4 = \frac{d\left(y^4\right)}{dy}\frac{dy}{dx} = 4y^3\frac{dy}{dx}.$$

Hence, equation (6.1) *gives:*

$$4y^3\frac{dy}{dx} - \left(y(-\sin(x)) + \frac{dy}{dx}\cdot\cos(x)\right) - 4x^3 = 0.$$

Grouping all the terms involving $\frac{dy}{dx}$ together, we have:

$$\left(4y^3 - \cos x\right)\frac{dy}{dx} = 4x^3 - y\sin x$$

Hence,

$$\frac{dy}{dx} = \frac{4x^3 - y\sin x}{4y^3 - \cos x}$$

The tangent line to the curve C at the point $(\pi/2, -\pi/2)$ is equal to:

$$\left.\frac{dy}{dx}\right|_{(\pi/2,-\pi/2)} = \frac{4(\pi/2)^3 + \pi/2}{-4(\pi/2)^3}$$

Hence, the equation of the tangent line is:

$$y = \left(\frac{4(\pi/2)^3 + \pi/2}{-4(\pi/2)^3}\right)(x - \pi/2) - \pi/2$$

Theorem 6.4. *Let f be an injective function differentiable at $x = c$. If $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$, with:*

$$\left(f^{-1}\right)'(f(c)) = \frac{1}{f'(c)} \ .$$

Equivalently, for any $y \in \mathrm{Range}(f)$, if f is differentiable at $x = f^{-1}(y)$, and $f'(f^{-1}(y)) \neq 0$, then:

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(f^{-1}(y))} \ .$$

Example 6.5. *Consider the injective function:*

$$f : [-\pi/2, \pi/2] \longrightarrow \mathbb{R},$$
$$f(x) = \sin x, \quad x \in [-\pi/2, \pi/2].$$

The inverse of f is:

$$f^{-1} = \arcsin : [-1, 1] \longrightarrow [-\pi/2, \pi/2].$$

Consider any $y \in (-1, 1)$. We have $y = f(x) = \sin(x)$ for a unique $x = \arcsin y$ in $(-\pi/2, \pi/2)$. Since $x \in (-\pi/2, \pi/2)$, we have $f'(x) = \cos(x) \neq 0$.

Hence, by Theorem 6.4 , $(f^{-1})'(y)$ exists, with:

$$\left(f^{-1}\right)'(y) = \left(f^{-1}\right)'(f(x)) = \frac{1}{f'(x)} = \frac{1}{\cos x} \ .$$

By the Pythagorean Theorem, we know that:

$$\cos x = \pm\sqrt{1 - \sin^2 x} \ .$$

Moreover, since $x \in (-\pi/2, \pi/2)$, we have $\cos x > 0$, so:

$$\cos x = +\sqrt{1 - \sin^2 x} = \sqrt{1 - \sin^2(\arcsin(y))} = \sqrt{1 - y^2}.$$

In conclusion, for $y \in (-1, 1)$, we have:

$$\arcsin' y = \left(f^{-1}\right)'(y) = \frac{1}{\sqrt{1 - y^2}} \ .$$

Example 6.6. *Similary, we can find the derivative of* arccos *as follows:*

The function arccos *is the inverse function* g^{-1} *of the following injective function:*

$$g(x) = \cos x, \quad x \in [0, \pi].$$

For any $y \in (-1, 1)$*, we have* $g^{-1}(y) \in (0, \pi)$*, so* $g'(g^{-1}(y)) = -\sin(\arccos(y)) \neq 0$.

Hence, by Theorem 6.4 , the function g^{-1} *is differentiable at* $y \in (-1, 1)$*, with:*

$$(g^{-1})'(y) = \frac{1}{g'(g^{-1}(y))} = \frac{1}{-\sin(\arccos(y))}.$$

By the Pythagorean Theorem, $\sin x = \pm\sqrt{1 - \cos^2(x)}$*. Since* $\arccos(y) \in (0, \pi)$ *for* $y \in (-1, 1)$*, we have:*

$$\sin(\arccos(y)) = +\sqrt{1 - \cos^2(\arccos(y))} = \sqrt{1 - y^2}.$$

Hence,

$$\arccos' y = (g^{-1})'(y) = -\frac{1}{\sqrt{1 - y^2}}.$$

6.2 WeBWorK

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6.3 Higher Order Derivatives

Let f be a function.

Its derivative f' is often called the **first derivative** of f.

The derivative of f', denoted by f'', is called the **second derivative** of f.

If $f''(c)$ exists, we say that f is **twice differentiable** at c.

For $n \in \mathbb{N}$, the n-th derivative of f, denoted by $f^{(n)}$ is defined as the derivative of the $(n-1)$-st derivative of f.

If $f^{(n)}(c)$ exists, we say that f is n times differentiable at c.

We sometimes consider f to be the "zero"-th derivative of itself, i.e. $f^{(0)} := f$.

In the Leibniz notation, we have:

$$f^{(n)}(x) = \underbrace{\frac{d}{dx}\frac{d}{dx}\cdots\frac{d}{dx}}_{n \text{ times}} f,$$

which is customarily written as:

$$\frac{d^n f}{dx^n} \quad .$$

Example 6.7. *Consider the curve:*

$$x^2 + y^2 = 1$$

Find $\frac{d^2y}{dx^2}$.

Solution. *Applying implicit differentiation, we have:*

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}1$$

$$2x + 2y\frac{dy}{dx} = 0 \tag{6.2}$$

This shows that:

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Applying implicit differentiation to equation (6.2)*, we have:*

$$\frac{d}{dx}\left(2x + 2y\frac{dy}{dx}\right) = \frac{d}{dx}0$$

$$2 + 2\left(y\frac{d^2y}{dx^2} + \frac{dy}{dx}\frac{dy}{dx}\right) = 0$$

It follows that:

$$\begin{aligned} y\frac{d^2y}{dx^2} &= -1 - \left(\frac{dy}{dx}\right)^2 \\ &= -1 - \frac{x^2}{y^2} \\ &= -\left(\frac{x^2+y^2}{y^2}\right) = -\left(\frac{1}{y^2}\right) \end{aligned}$$

Hence,

$$\frac{d^2y}{dx^2} = -\left(\frac{1}{y^3}\right)$$

Example 6.8. *Let:*

$$f(x) = \begin{cases} x^4 \sin\left(\frac{1}{x}\right) & \textit{if } x \neq 0; \\ 0 & \textit{if } x = 0. \end{cases}$$

Find $f''(0)$, if it exists.

Solution. *For $x \neq 0$, we have:*

$$\begin{aligned} f'(x) &= \frac{d}{dx}x^4 \sin(1/x) \\ &= 4x^3 \sin(1/x) + x^4 \cos(1/x) \cdot (-x^{-2}) \\ &= 4x^3 \sin(1/x) - x^2 \cos(1/x) \\ &= x^2(4x \sin(1/x) - \cos(1/x)) \end{aligned}$$

By the limit definition of the derivative, we have:

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{h^4 \sin(1/h) - 0}{h} \\ &= \lim_{h\to 0} h^3 \sin(1/h) = 0 \quad (\textit{by Sandwich Theorem}) \end{aligned}$$

Hence,

$$f'(x) = \begin{cases} x^2(4x\sin(1/x) - \cos(1/x)), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

By definition:

$$f''(0) = (f')'(0) = \lim_{h\to 0} \frac{f'(0+h) - f'(0)}{h}.$$

Hence,

$$\begin{aligned} f''(0) &= \lim_{h\to 0} \frac{h^2(4h\sin(1/h) - \cos(1/h)) - 0}{h} \\ &= \lim_{h\to 0} h(4h\sin(1/h) - \cos(1/h)) \\ &= 0 \ \ (\textit{again by Sandwich Theorem}). \end{aligned}$$

Theorem 6.9 (General Leibniz Rule)**.** *Let $n \in \mathbb{N}$. Given any functions f, g which are n times differentiable at c, their product fg is also n times differentiable at c, with:*

$$(fg)^{(n)}(c) = \sum_{k=0}^{n} C_k^n f^{(k)}(c) g^{(n-k)}(c)$$

Notice that when $n = 1$ this rule is simply the product rule we have introduced before.

Example 6.10. *Consider $h(x) = x^2\sin(x)$. Then, $h = fg$, where $f(x) = x^2$ and $g(x) = \sin(x)$.*

We have:

$$f'(x) = 2x, \quad f''(x) = 2, \quad f^{(3)}(x) = 0.$$
$$g'(x) = \cos(x), \quad g''(x) = -\sin x, \quad g^{(3)}(x) = -\cos(x).$$

Hence, by the General Leibniz Rule, the first, second and third derivatives of h may be computed as follows:

$$\begin{aligned} h'(x) &= fg'(x) + f'g(x) \\ &= x^2\cos(x) + 2x\sin(x) \end{aligned}$$

$$\begin{aligned} h''(x) &= fg''(x) + 2f'g'(x) + f''g(x) \\ &= x^2(-\sin(x)) + 2(2x)\cos(x) + 2\sin(x) \end{aligned}$$

$$\begin{aligned} h^{(3)}(x) &= fg^{(3)}(x) + 3f'g''(x) + 3f''g'(x) + f^{(3)}g(x) \\ &= x^2(-\cos(x)) + 3(2x)(-\sin(x)) + 3(2)\cos(x) + 0\cdot\sin(x) \\ &= -x^2\cos(x) - 6x\sin(x) + 6\cos(x) \end{aligned}$$

6.4 WeBWorK

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