Math 1010 Week 4

Limits, Continuity

4.1 More Limit Identities

Example 4.1. *Find:*

- $\lim_{x \to 0^+} \sin\left(\frac{1}{x}\right)$
- $\lim_{x \to 0^+} x \sin\left(\frac{1}{x}\right)$

Definition 4.2. *For each $x \in \mathbb{R}$, we let:*

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

It is known that:

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n.$$

Theorem 4.3.

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$$

Corollary 4.4.

$$\lim_{x \to \infty} \left(1 - \frac{1}{x}\right)^x = \lim_{x \to 0} (1 - x)^{\frac{1}{x}} = \frac{1}{e}$$

For all $a \in \mathbb{R}$,

$$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a$$

Exercise 4.5. *Find:*

$$\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x$$

Theorem 4.6. *For all $n \in \{1, 2, 3, \ldots\}$, we have:*

$$\lim_{x\to\infty}\frac{x^n}{e^x} = 0.$$

Corollary 4.7. *For all $n \in \{1, 2, 3, \ldots\}$, and $b > 1$, we have:*

$$\lim_{x\to\infty}\frac{x^n}{b^x} = 0.$$

Fact 4.8.

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1.$$

From this may be further deduced that:

$$\lim_{t\to 0}\frac{e^t-1}{t} = 1,$$

by applying a change of variable:

$$x = e^t - 1.$$

4.2 Continuity

Definition 4.9. *A function $f : A \longrightarrow \mathbb{R}$ is said to be* **continuous** *at $c \in A$ if:*

$$\lim_{x\to c} f(x) = f(c).$$

A function is said to be **continuous** *if it is continuous at every point in its domain.*

Should c be an endpoint in the domain of f, the continuity of f at c is defined in terms of a one-sided limit. That is, right limit if c is a left endpoint, and left limit if c is a right endpoint. Hence, the function:

$$f(x) = \sqrt{x}$$

is continuous at $x = 0$, since $\mathrm{Domain}(f) = [0, \infty)$, and:

$$\lim_{x\to 0^+} f(x) = \ 0 \ = f(0).$$

The following "elementary functions" are continuous at every element in their domains:

$$f(x) = x, \frac{1}{x}, \sin x, \cos x, \tan x, e^x, \ln x, \arcsin x, \arccos x, \arctan x$$

Due to the laws of sum/difference/product/quotient for limits, the sum/difference/product/quotient of continuous functions is also continous.

In particular, polynomials and rational functions are all continuous on their domains.

Theorem 4.10. *For functions g, f with the property that $\lim\limits_{x\to a} g(x)$ exists and f is continuous at $\lim_{x\to a} g(x)$, we have:*

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right).$$

Example 4.11. *It follows from this theorem that:*

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

It also follows from the previous theorem that any composite of continuous functions is continuous.

Example 4.12. *The following functions are all continuous, since they are the sums, differences, products, quotients, or composites of other continuous functions:*

$$f(x) = \frac{e^{\cos\left(\frac{1}{x}\right)}}{x^7 - 9x^2 + 23}$$
$$g(x) = \frac{1}{\arctan x} - \sqrt[3]{\log_5\left(2^x + 1\right)}$$
$$h(x) = \sin\left(x^{-3} + \left(\cos\left(e^{x^2} + 1\right)\right)\right)$$

Example 4.13. *The following functions are continuous at every point on the real line:*

- $$g(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0; \\ 1, & x = 0; \end{cases}$$

- $$f(x) = \begin{cases} x^2 \cos\left(\dfrac{1}{e^x - 1}\right), & x \neq 0; \\ 0, & x = 0; \end{cases}$$

Exercise 4.14. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies:*

- *$f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$;*
- *$f(x)$ is continuous at $x = 0$ and $f(0) \neq 0$.*

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1. *Show that $f(0) = 1$.*
2. *Show that $f(x)$ is continuous on $\mathbb{R}$.*

4.2.1 WeBWorK

1. **WeBWorK**
2. **WeBWorK**
3. **WeBWorK**
4. **WeBWorK**

4.2.2 Further Properties of Continuous Functions

Theorem 4.15 (Intermediate Value Theorem IVT)**.** *If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous, then f attains every value between $f(a)$ and $f(b)$. In other words, for any $y \in \mathbb{R}$ between the values of $f(a)$ and $f(b)$, there exists $c \in [a, b]$ such that $f(c) = y$.*

Exercise 4.16.

- *Show that $f(x) = x^5 + x^2 - 10 = 0$ has a real root between $x = 1$ and $x = 2$.*
- *Show that the range of $f(x) = e^x - \sqrt{x}$ contains $[1, \infty)$.*

Theorem 4.17 (Extreme Value Theorem)**.** *If f is a continuous function defined on a closed interval $[a, b]$, then it attains both a maximum value and a minimum value on $[a, b]$.*