Math 1010 Week 12

Indefinite Integrals, Reduction Formulas, Partial Fractions, t-Substitution

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12.1 Reduction Formulas

Let $n \in \mathbb{N}$.

Example 12.1.

$$\underbrace{\int x^n e^x \, dx}_{I_n} = x^n e^x - n \underbrace{\int x^{n-1} e^x \, dx}_{I_{n-1}} .$$

Example 12.2. *For $n \geq 2$,*

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$

Let $U = \cos^{n-1} x$, $dV = \cos x \, dx$. Then:

$$dU = -(n-1) \cos^{n-2} x \sin x \, dx, \quad V = \sin x.$$

It follows from Section 10.8 () that:

$$\begin{aligned}
\int U \, dV = UV - \int V \, dU \\
&= \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x \, dx \\
&= \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx
\end{aligned}$$

Hence:

$$(1 + (n-1)) \int \cos^n x\, dx$$
$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x\, dx.$$

Dividing both sides of the equation by n, we obtain:

$$\int \cos^n x\, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x\, dx.$$

Example 12.3. *For $n \geq 2$,*

$$\int \sin^n x\, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x\, dx.$$

Example 12.4. *For $n \geq 3$,*

$$\int \sec^n x\, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx.$$

Example 12.5.

$$\int (\ln x)^n\, dx = x(\ln x)^n - n \int (\ln x)^{n-1}\, dx.$$

12.2 WeBWorK

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12.3 Partial Fractions

Definition 12.6. *A rational function $\frac{r}{s}$, where r, s are polynomials, is said to be* **proper** *if:*

$$\deg r < \deg s.$$

By performing long division of polynomials, any rational function $\frac{p}{q}$, where p, q are polynomials, may be expressed in the form:

$$\frac{p}{q} = g + \frac{r}{s},$$

where g is a polynomial, and $\frac{r}{s}$ is a proper rational function. Let $\frac{r}{s}$ be a proper rational function. Factor s as a product of powers of distinct irreducible factors:

$$s = \cdots (x-a)^m \cdots (\underbrace{x^2+bx+c}_{\text{irreducible i.e. } b^2-4c<0})^n \cdots .$$

Then:

Fact 12.7. *The proper rational function $\frac{r}{s}$ may be written as a sum of rational functions as follows:*

$$\begin{aligned}\frac{r}{s} = \cdots \\ &+ \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_m}{(x-a)^m} + \cdots \\ &+ \frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(x^2+bx+c)^n} \\ &+ \cdots ,\end{aligned}$$

where the A_i, B_i, C_i are constants.

Example 12.8. $\int \frac{x^3-x-2}{x^2-2x}\,dx$

Performing long division for polynomials, we have:

$$\begin{aligned}\int \frac{(x^3-x-2)}{x^2-2x}dx &= \int (x+2)dx + \int \frac{3x-2}{x^2-2x}dx \\ &= \frac{1}{2}x^2 + 2x + \int \frac{3x-2}{x^2-2x}dx.\end{aligned}$$

To evaluate:

$$\int \frac{3x-2}{x^2-2x}dx,$$

we first observe that the integrand is a proper rational function. Moreover, the denominator factors as follows:

$$x^2 - 2x = x(x-2).$$

Hence, by Fact 12.7 , we have:

$$\frac{3x-2}{x^2-2x} = \frac{A}{x} + \frac{B}{x-2},$$

for some constants A and B. Clearing denominators, we see that the equation above holds if and only if:

$$3x - 2 = A(x-2) + Bx. \qquad (*)$$

Letting $x = 2$, we have:

$$3 \cdot 2 - 2 = B \cdot 2,$$

which implies that $B = 2$. Similarly, letting $x = 0$ in equation $(*)$ gives:

$$-2 = -2A,$$

which implies that $A = 1$. Hence:

$$\begin{aligned} \int \frac{3x-2}{x^2-2x} dx &= \int \left(\frac{1}{x} + \frac{2}{x-2} \right) dx \\ &= \ln|x| + 2\ln|x-2| + C, \end{aligned}$$

where C represents an arbitrary constant.

We conclude that:

$$\int \frac{(x^3 - x - 2)}{x^2 - 2x} dx = \frac{1}{2}x^2 + 2x + \ln|x| + 2\ln|x-2| + C.$$

Example 12.9. $\int \frac{x}{(x^2+4)(x-3)}\, dx$

First we note that the integrand is a proper rational function.

The quadratic factor $x^2 + 4$ has discriminant $0^2 - 4 \cdot 4 < 0$, hence it is irreducible.

By Fact 12.7 , we have:

$$\frac{x}{(x^2+4)(x-3)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-3},$$

for some constants A, B and C. Clearing denominators, the equation above holds if and only if:

$$x = (Ax+B)(x-3) + C(x^2+4) \qquad (*)$$

Letting $x = 3$, we have:

$$3 = C \cdot 13,$$

which implies that $C = 3/13$.

Letting $x = 0$, *we have:*

$$0 = -3B + 4C,$$

which implies that $B = (4/3)C = 4/13$.

Finally, viewing each side of equation $(*)$ *as polynomials and comparing the coefficients of* x^2 *on each side, we have:*

$$0 = A + C,$$

which implies that $A = -C = -3/13$.

Hence:

$$\begin{aligned}
&\int \frac{x}{(x^2+4)(x-3)}\,dx \\
&= \frac{1}{13}\int \frac{-3x+4}{x^2+4}\,dx + \frac{3}{13}\int \frac{1}{x-3}\,dx \\
&= \frac{1}{13}\left(\frac{-3}{2}\int \frac{1}{x^2+4}\,d(x^2+4) + \int \frac{1}{(x/2)^2+1}\,dx \right. \\
&\quad \left. +3\int \frac{1}{x-3}\,dx\right) \\
&= \frac{1}{13}\left(\frac{-3}{2}\ln\left|x^2+4\right| + 2\arctan(x/2) + 3\ln|x-3|\right) + D,
\end{aligned}$$

where D *represents an arbitrary constant.*

Example 12.10. $\displaystyle\int \frac{x^3}{(x^2+x+1)(x-3)^2}\,dx$

First, we observe that:

$$\frac{x^3}{(x^2+x+1)(x-3)^2}$$

is a proper rational function. Moreover, since the discriminant of x^2+x+1 *is* $1^2 - 4 < 0$, *this quadratic factor is irreducible. So, there exist constants* A, B, C, D *such that:*

$$\frac{x^3}{(x^2+x+1)(x-3)^2} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-3} + \frac{D}{(x-3)^2}.$$

The equation above holds if and only if:

$$\begin{aligned} x^3 &= (Ax+B)(x-3)^2 + C(x^2+x+1)(x-3) \\ &\quad + D(x^2+x+1). \end{aligned} \qquad (*)$$

Letting $x = 3$, we have:

$$27 = 13D.$$

So, $D = 27/13$.

To find A, B and C, we view each side of the equation $(*)$ as polynomials, then compare the coefficients of the x^3, x^2, x and constant terms respectively:

$$x^3: \qquad 1 = A + C \tag{12.1}$$

$$x^2: \qquad 0 = -6A + B - 2C + 27/13 \tag{12.2}$$

$$x: \qquad 0 = 9A - 6B - 2C + 27/13 \tag{12.3}$$

$$1: \qquad 0 = 9B - 3C + 27/13 \tag{12.4}$$

Subtracting equation (12.2) *from equation* (12.3)*, we have:*

$$0 = 15A - 7B,$$

which implies that $B = 15A/7$. Combining this with equation (12.1)*, we have:*

$$B = 15(1 - C)/7 = 15/7 - 15C/7.$$

It now follows from equation (12.4) *that:*

$$0 = 135/7 - 135C/7 - 3C + 27/13.$$

Hence:

$$C = \frac{162}{169}$$
$$B = \frac{15}{169}$$
$$A = \frac{7}{169}$$
$$D = \frac{27}{13}.$$

We have:

$$\int \frac{x^3}{(x^2 + x + 1)(x - 3)^2}\, dx = \int \left[\frac{7x + 15}{169\,(x^2 + x + 1)} + \frac{162}{169\,(x - 3)} + \frac{27}{13\,(x - 3)^2}\right] dx$$

$$= \int \frac{7x + 15}{169\,(x^2 + x + 1)} dx + \frac{162}{169} \int \frac{1}{(x - 3)} dx + \frac{27}{13} \int \frac{1}{(x - 3)^2} dx$$

To evaluate $\int \frac{7x+15}{169(x^2+x+1)}dx$*, we first rewrite the integral as follows:*

$$\int \frac{7x+15}{169\,(x^2+x+1)}dx = \frac{1}{169}\int \frac{7x+7/2-7/2+15}{x^2+x+1}dx$$

$$= \frac{1}{169}\left[\frac{7}{2}\underbrace{\int \frac{2x+1}{x^2+x+1}dx}_{\int \frac{1}{x^2+x+1}d(x^2+x+1)} + \frac{23}{2}\underbrace{\int \frac{1}{(x+1/2)^2+3/4}dx}_{\frac{4}{3}\int \frac{1}{((2x+1)/\sqrt{3})^2+1}dx}\right]$$

$$= \frac{7}{338}\ln\left|x^2+x+1\right| + \frac{23\cdot 2}{169\cdot 3}\frac{\sqrt{3}}{2}\arctan\left((2x+1)/\sqrt{3}\right) + E$$

$$= \frac{7}{338}\ln\left|x^2+x+1\right| + \frac{23}{169\sqrt{3}}\arctan\left((2x+1)/\sqrt{3}\right) + E,$$

where E *represents an arbitrary constant.*

It now follows that:

$$\int \frac{x^3}{(x^2+x+1)(x-3)^2}\,dx = \frac{7}{338}\ln\left|x^2+x+1\right| + \frac{23}{169\sqrt{3}}\arctan\left((2x+1)/\sqrt{3}\right) + \frac{162}{169}\ln|x-3| - \frac{27}{13}\frac{1}{x-3} + E.$$

Example 12.11. $\int \frac{8x^2}{x^4+4}\,dx$

12.4 WeBWorK

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12. **WeBWorK**
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14. **WeBWorK**
15. **WeBWorK**

12.5 How Does Partial Fractions Decomposition Work?

This section is optional. You don't have to study it for Math 1010.

Theorem 12.12 (Unique Factorization of Real Polynomials)**.** *Given any polynomial $f \in \mathbb{R}[x]$, that is:*

$$f = a_0 + a_1 x + ... + a_n x^n, \quad a_i \in \mathbb{R},$$

There are distinct irreducible polynomials, $p_1, p_2, \ldots, p_l$ in $\mathbb{R}[x]$, of degree at most 2, such that:

$$f = p_1^{n_1} p_2^{n_2} \cdots p_l^{n_l}$$

for some positive integers $n_1, n_2, \ldots, n_l$. Up to ordering of the irreducible factors, this factorization is unique.

Theorem 12.13 (Bézout's Identity)**.** *If f and g are two irreducible polynomials in $\mathbb{R}[x]$ with no common factors, then there exist $a, b \in \mathbb{R}[x]$ such that:*

$$af + bg = 1$$

Suppose we have a rational function $\frac{p}{q}$, where $p, q \in \mathbb{R}[x]$ have no common factors, and $\deg p < \deg q$.

By Unique Factorization of Real Polynomials , there are distinct irreducible polynomials $q_1, q_2, \ldots, q_l$, of degree at most 2, such that:

$$q = q_1^{n_1} q_2^{n_2} \cdots q_l^{n_l},$$

for some positive integers $n_1, n_2, \ldots, n_l$.

Since the polynomial $q_1^{n_1}$ has no common factors with $q_2^{n_2} \ldots q_l^{n_l}$, by Bézout's Identity there exist polynomials f, g such that:

$$f \cdot (q_2^{n_2} \cdots q_l^{n_l}) + g q_1^{n_1} = 1.$$

Hence,

$$\begin{aligned}\frac{p}{q} &= \frac{p \cdot 1}{q} \\ &= \frac{p(f q_2^{n_2} \cdots q_l^{n_l} + g q_1^{n_1})}{q_1^{n_1} q_2^{n_2} \cdots q_1^{n_l}} \\ &= \frac{pf}{q_1^{n_1}} + \frac{pg}{q_2^{n_2} \cdots q_l^{n_l}}\end{aligned}$$

Consider now the term: $\dfrac{pf}{q_1^{n_1}}$. By the Divison Algorithm for real polynomials, we have:

$$pf = aq_1 + r$$

for some real polynomials a, r such that $\deg r < \deg q_1$. Hence,

$$\frac{pf}{q_1^{n_1}} = \frac{aq_1 + r}{q_1^{n_1}} = \frac{a}{q_1^{n_1 - 1}} + \frac{r}{q_1^{n_1}}$$

By the same reasoning, we have:

$$\frac{a}{q_1^{n_1-1}} = \frac{b}{q_1^{n_1-2}} + \frac{s}{q_1^{n_1-1}}$$

for some polynomials b, s such that $\deg s < \deg q_1$.

Repeating this process, eventually we have:

$$\frac{pf}{q_1^{n_1}} = \frac{r_1}{q_1} + \frac{r_2}{q_1^2} + \cdots + \frac{r_{n_1}}{q_1^{n_1}} + a_1,$$

where $\deg r_i < \deg q_1$, and a_1 is some polynomial.

We now have:

$$\frac{p}{g} = \frac{r_1}{q_1} + \frac{r_2}{q_1^2} + \cdots + \frac{r_{n_1}}{q_1^{n_1}} + a_1 + \frac{pg}{q_2^{n_2} \cdots q_l^{n_l}}.$$

Repeating the process for the term: $\dfrac{pg}{q_2^{n_2} \cdots q_l^{n_l}}$, and then for all subsequent resulting terms of similar forms, we have:

$$\frac{p}{q} = \sum_{k=1}^{l} \sum_{j=1}^{n_k} \frac{r_{kj}}{q_k^j} + h, \tag{12.5}$$

where $\deg r_{kj} < \deg q_k$, and h is some polynomial in $\mathbb{R}[x]$.

We claim that $h = 0$.

Multiplying both sides of equation (12.5) by the polynomial q, we have:

$$p = \sum_{k=1}^{l} \sum_{j=1}^{n_k} r_{kj} \cdot \frac{q}{q_k^j} + hq \tag{12.6}$$

Since every q_k^j in the sum divides q, each $\frac{q}{q_k^j}$ is a polynomial. So, the equation above is an equality between polynomials.

By assumption, $\deg p < \deg q$. On the other hand, each term:

$$r_{kj} \cdot \frac{q}{q_k^j}$$

has degree strictly less than q, since $\deg r_{kj} < \deg q_k$.

So, if $h \neq 0$, then the right-hand side of equation (12.6) has degree $\deg h + \deg q \geq \deg q > \deg p$, contradicting the equality of the two sides.

Hence, $h = 0$. It follows that:

$$\frac{p}{q} = \sum_{k=1}^{l} \sum_{j=1}^{n_k} \frac{r_{kj}}{q_k^j}$$

12.6 t-Substitution

Example 12.14. *Evaluate:*

$$\int \frac{1}{1 + 2\cos x}\, dx$$

Let:

$$t = \tan \frac{x}{2}.$$

(Here, we are assuming that $x \in (-\pi, \pi)$).

Then,

$$x = 2 \arctan t,$$
$$dx = \frac{2}{1+t^2} dt$$

Moreover,

by the double-angle formula for the sine function, we have:

$$\begin{aligned}\sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2}\\ &= 2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cos^2\frac{x}{2}\\ &= \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}}\\ &= \frac{2t}{1+t^2}\end{aligned}$$

Similarly, by the double-angle formula for the cosine function, we have:

$$\begin{aligned}\cos x &= 1-2\sin^2\frac{x}{2}\\ &= 1-2\tan^2\frac{x}{2}\cos^2\frac{x}{2}\\ &= 1-\frac{2\tan^2\frac{x}{2}}{\sec^2\frac{x}{2}}\\ &= 1-\frac{2t^2}{1+t^2}\\ &= \frac{1-t^2}{1+t^2}\end{aligned}$$

We have:

$$\begin{aligned}\int\frac{1}{1+2\cos x}\,dx &= \int\frac{1}{1+2\left(\frac{1-t^2}{1+t^2}\right)}\frac{2}{1+t^2}\,dt\\ &= \int\frac{2}{3-t^2}\,dt\\ &= \frac{1}{\sqrt{3}}\int\left(\frac{1}{\sqrt{3}+t}+\frac{1}{\sqrt{3}-t}\right)dt\\ &= \frac{1}{\sqrt{3}}\left(\ln\left|\sqrt{3}+t\right|-\ln\left|\sqrt{3}-t\right|\right)+C\\ &= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}+\tan\frac{x}{2}}{\sqrt{3}-\tan\frac{x}{2}}\right|+C,\end{aligned}$$

where C is an arbitrary constant.

Example 12.15. *Evaluate:*

$$\int\frac{1}{1+\sin x+\cos x}dx$$

Let $t = \tan\frac{x}{2}$. *Then:*

$$\begin{aligned} dx &= \frac{2}{1+t^2}dt \\ \sin x &= \frac{2}{1+t^2} \\ \cos x &= \frac{1-t^2}{1+t^2} \end{aligned}$$

$$\begin{aligned} \int \frac{1}{1+\sin x+\cos x}dx = \int \frac{\frac{2}{1+t^2}dt}{1+\frac{2}{1+t^2}+\frac{1-t^2}{1+t^2}} \\ \int \frac{2dt}{2+2t} = \int \frac{1}{1+t}dt \\ \ln|1+t|+C \\ \ln\left|1+\tan\frac{x}{2}\right|+C \\ \ln\left|1+\frac{\sin x}{1+\cos x}\right|+C, \end{aligned}$$

where C *is an arbitrary constant.*