Math 1010 Week 1

Sequences

1.1 Sequences and Limits

A **sequence** is an ordered list of numbers:

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

Common notations:

$$\{a_n\},\ \{a_n\}_{n\in\mathbb{N}},\ \{a_n\}_{n=1}^{\infty}$$

Example 1.1.

- $$a_n = \sqrt{n}\,, \quad n \in \mathbb{N}$$
  $$\{a_n\}_{n\in\mathbb{N}} = \{1, \sqrt{2}, \sqrt{3}, \ldots\}.$$

- $$b_n = (-1)^{n+1}\frac{1}{n}\,, \quad n \in \mathbb{N}$$
  $$\{b_n\} = \left\{1, -\frac{1}{2}\,, \frac{1}{3}\,, -\frac{1}{4}\,, \ldots\right\}.$$

- **Fibonacci Sequence**

$$a_1 = 1, a_2 = 1$$

$$a_n = a_{n-2} + a_{n-1} \mathit{for} n \geq 3.$$

$$\{a_n\} = \{1, 1, 2, 3, 5, 8, 13, \ldots\}$$

In this case we say that the sequence $\{a_n\}$ is defined **recursively** .

Sometimes, the terms a_n of a sequence approach a single value L as n tends to infinity.

Definition 1.2. *We say that the* **limit** *of a sequence $\{a_n\}$ is equal to L if for all real numbers $\varepsilon > 0$ the exists a number $N > 0$ such that $|a_n - L| < \varepsilon$ for all $n > N$.*

If such a number L exists, we say that: $\{a_n\}$ **converges** to L, and write:

$$\lim_{n\to\infty} a_n = L.$$

If no such L exists, we say that $\{a_n\}$ **diverges** .

If the values of a_n increase (resp. decrease) without bound, we say that $\{a_n\}$ diverges to ∞ (resp. $-\infty$), and write:

$$\lim_{n\to\infty} a_n = \infty \quad \text{(resp. } -\infty).$$

Exercise 1.3. *1.* **WeBWorK**

2. **WeBWorK**

3. **WeBWorK**

4. **WeBWorK**

1.1.1 Useful Properties

- **Constant sequence**

  If $a_n = c$ for all n, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c = c$.

- **Sum/Difference rule**

  If both $\{a_n\}$ and $\{b_n\}$ converge, then:

$$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n.$$

- **Product Rule**

  If both $\{a_n\}$ and $\{b_n\}$ converge, then:

$$\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right).$$

- **Quotient Rule**

  If both $\{a_n\}$ and $\{b_n\}$ converge, and $\lim_{n\to\infty} b_n \neq 0$, then:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

- $$\lim_{n\to\infty} \frac{1}{n} = 0.$$

- In general, if $\lim_{n\to\infty} a_n = +\infty$ or $\lim_{n\to\infty} a_n = -\infty$, we have:

$$\lim_{n\to\infty} \frac{1}{a_n} = 0.$$

1.1.2 Examples

- $\lim_{n\to\infty} \frac{3n^2 - 2n + 7}{2n^2 + 3}$

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \cdot \frac{3n^2 - 2n + 7}{2n^2 + 3} \\
&= \lim_{n\to\infty} \frac{3 - \frac{2}{n} + \frac{7}{n^2}}{2 + \frac{3}{n^2}} \\
&= \frac{3}{2}.
\end{aligned}$$

- $\lim_{n\to\infty} \frac{-3n^2}{\sqrt[3]{27n^6 - 5n + 1}}$

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{-3n^2}{n^2 \sqrt[3]{27 - \frac{5}{n^5} + \frac{1}{n^6}}} \\
&= \lim_{n\to\infty} \frac{-3}{\sqrt[3]{27 - \frac{5}{n^5} + \frac{1}{n^6}}} \\
&= \frac{-3}{\sqrt[3]{27}} = -1.
\end{aligned}$$

- $\lim_{n\to\infty} \sqrt{4n^2 + n} - \sqrt{4n^2 - 1}$

$$
\begin{aligned}
&= \lim_{n\to\infty} \left(\sqrt{4n^2+n} - \sqrt{4n^2-1}\right) \cdot \frac{\left(\sqrt{4n^2+n} + \sqrt{4n^2-1}\right)}{\left(\sqrt{4n^2+n} + \sqrt{4n^2-1}\right)} \\
&= \lim_{n\to\infty} \frac{(4n^2+n) - (4n^2-1)}{\left(\sqrt{4n^2+n} + \sqrt{4n^2-1}\right)} \\
&= \lim_{n\to\infty} \frac{n+1}{\sqrt{4n^2+n} + \sqrt{4n^2-1}} \\
&= \lim_{n\to\infty} \frac{n+1}{n\left(\sqrt{4+\frac{1}{n}} + \sqrt{4-\frac{1}{n^2}}\right)} \\
&= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{\left(\sqrt{4+\frac{1}{n}} + \sqrt{4-\frac{1}{n^2}}\right)} \\
&= \frac{1}{4}.
\end{aligned}
$$

Exercise 1.4. • **WeBWorK**

1.1.3 Monotonic Sequences

Definition 1.5. *A sequence $\{a_n\}$ is said to be:*

- **increasing** *if $a_{n+1} \geq a_n$ for all n,*
- **decreasing** *if $a_{n+1} \leq a_n$ for all n.*

A sequence is said to be **monotonic** *if it is either increasing or decreasing.*

Theorem 1.6 (Monotone Convergence Theorem)**.** *If $\{a_n\}$ is either:*
increasing (i.e. $a_{n+1} \geq a_n$ for all n) and bounded above (i.e. There exists a number M such that $a_n \leq M$ for all n.), or
decreasing (i.e. $a_{n+1} \leq a_n$ for all n) and bounded below (i.e. There exists a number M such that $a_n \geq M$ for all n.), then $\{a_n\}$ converges.

Moreover,
if $\{a_n\}$ is increasing and $a_n \leq M$ for all n, then $\lim_{n\to\infty} a_n \leq M$.
If $\{a_n\}$ is decreasing and $a_n \geq M$ for all n, then $\lim_{n\to\infty} a_n \geq M$.

Example 1.7. *Let $\{a_n\}$ be a sequence of real numbers, which is defined by*

$$a_1 = 1 \qquad \textit{and} \qquad a_n = \frac{12a_{n-1}+12}{a_{n-1}+13} \textit{ for } n > 1.$$

1. *Prove that* $0 \leq a_n \leq 3$. *(Hint: Perhaps* **mathematical induction** *could be useful here.)*

2. *Prove that* $\{a_n\}$ *converges (i.e.* $\lim_{n\to\infty} a_n$ *exists), then find its limit.*

Solution. *1. First, we show that* $a_n \geq 0$ *for all* $n \in \mathbb{N}$.

Base Step : By definition, $a_1 = 1 \geq 0$.

Inductive Step : Suppose $a_n \geq 0$ *for some* $n \in \mathbb{N}$. *We want to show that* $a_{n+1} \geq 0$ *also.*

By the definition of the sequence, we have:

$$a_{n+1} = \frac{12a_n + 12}{a_n + 13}.$$

By the **induction hypothesis** *, i.e.* $a_n \geq 0$, *we have:*

$$a_n + 13 > 0 \quad \textit{and} \quad 12a_n + 12 \geq 0.$$

Hence, $a_{n+1} \geq 0$.

It now follows from the principle of mathematical induction that $a_n \geq 0$ *for all* $n \in \mathbb{N}$.

Similary, to show that $a_n \leq 3$, *we first observe that by definition* $a_1 = 1 \leq 3$. *Whenever* $a_n \leq 3$, *we have:*

$$\begin{aligned} 3 - a_{n+1} &= 3 - \frac{12a_n + 12}{a_n + 13} \\ &= \frac{3a_n + 39 - 12a_n - 12}{a_n + 13} \\ &= \frac{9(3 - a_n)}{a_n + 13} \geq 0, \end{aligned}$$

which implies that $a_{n+1} \leq 3$ *also. Hence, by mathematical induction we conclude that* $a_n \leq 3$ *for all* $n \in \mathbb{N}$.

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2. Observe that for all $n \in \mathbb{N}$, we have:

$$\begin{aligned}
a_{n+1} - a_n &= \frac{12a_n + 12}{a_n + 13} - a_n \\
&= \frac{12a_n + 12 - a_n^2 - 13a_n}{a_n + 13} \\
&= -\frac{a_n^2 + a_n - 12}{a_n + 13} \\
&= -\frac{(a_n - 3)(a_n + 4)}{a_n + 13} \\
&\geq 0,
\end{aligned}$$

since $0 \leq a_n \leq 3$, as shown in Part 1.

This shows that $\{a_n\}$ is an increasing sequence bounded above by 3. Hence, the limit $L = \lim_{n\to\infty} a_n$ exists, by the Monotone Convergence Theorem.

To find L, we take the limit as $n \to \infty$ of both sides of the equation:

$$a_n = \frac{12a_{n-1} + 12}{a_{n-1} + 13}.$$

That is:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{12a_{n-1} + 12}{a_{n-1} + 13},$$

which gives:

$$L = \frac{12L + 12}{L + 13},$$

since $\lim_{n\to\infty} a_{n-1} = \lim_{n\to\infty} a_n = L$.

The equation above implies that:

$$L^2 + L - 12 = 0,$$

which gives $L = 3$ or $L = -4$. Since the sequence $\{a_n\}$ is bounded below by 0, we may eliminate the case $L = -4$.

We conclude that:

$$\lim_{n\to\infty} a_n = 3.$$

1.1.4 Sandwich Theorem

Theorem 1.8 (Sandwich Theorem for Sequences)**.** *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences such that:*

$$a_n \leq b_n \leq c_n$$

for all n *sufficiently large. If*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then $\lim_{n\to\infty} b_n = L$ *also.*