MATH1050 Proof-writing Exercise 10

Advice.

- Study the Handouts *Image sets and pre-image sets*, *Image sets and pre-image sets under 'nice' real-valued functions of one real variable*, *Theoretical results involving image sets and pre-image sets*, *Characterization of surjectivity with image sets, pre-image sets*, *Relations, functions and 'well-defined-ness' for functions* before answering the questions.
- Besides the handouts mentioned above, Question (17) of Exercise 9 is also relevant.

1. Let $f : \mathbb{C}\backslash\{0\} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = \dfrac{i\bar{z}}{z}$ for any $z \in \mathbb{C}\backslash\{0\}$.

   Let $H = \{z \in \mathbb{C} : \mathsf{Re}(z) > 0\}$, and $S = \{w \in \mathbb{C} : |w| = 1\}$.

   (a) Prove that $f(H) \subset S\backslash\{-i\}$, with reference to the definition of *image sets*.

   (b) Prove that $S\backslash\{-i\} \subset f(H)$, with reference to the definition of *image sets*.

2. (a) Is the statement ($\sharp$) true? Justify your answer with reference to the definition of *pre-image sets*:

   ($\sharp$) *Let A, B be sets, and $f : A \longrightarrow B$ be a function. Let U, V be subsets of B. Suppose $U \subset V$. Then $f^{-1}(U) \subset f^{-1}(V)$.*

   (b) Is the statement ($\flat$) true? Justify your answer with reference to the definition of *pre-image sets*:

   ($\flat$) *Let A, B be sets, and $f : A \longrightarrow B$ be a function. Let U, V be subsets of B. Suppose $f^{-1}(U) \subset f^{-1}(V)$. Then $U \subset V$.*

3. (a) Is the statement ($\sharp$) true? Justify your answer with reference to the definitions of *image sets* and *pre-image sets*:

   ($\sharp$) *Let A, B be sets, and $f : A \longrightarrow B$ be a function. For any subset S of A, $S \subset f^{-1}(f(S))$.*

   (b) Is the statement ($\flat$) true? Justify your answer with reference to the definitions of *image sets* and *pre-image sets*:

   ($\flat$) *Let A, B be sets, and $f : A \longrightarrow B$ be a function. For any subset S of A, $f^{-1}(f(S)) \subset S$.*

4. Let $C = \{(x, y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R} \text{ and } 9x^2 + 16y^2 = 144\}$.

   (a) Let $A = [0, 4]$, $B = [0, 3]$, and $F = C \cap (A \times B)$. Define $f = (A, B, F)$.
   Verify that f is a function.

   (b) Let $A = [2, 3]$, $B = [-1, 4]$, and $F = C \cap (A \times B)$. Define $f = (A, B, F)$.
   Is f a function? Justify your answer.

   (c) Let $A = [1, 4]$, $B = [0, 5/2]$, and $F = C \cap (A \times B)$. Define $f = (A, B, F)$.
   Is f a function? Justify your answer.