MATH1050 Proof-writing Exercise 8 (Answers and selected solution)

1. (a) **Answer.**
   For any $z \in \mathbb{C}$, $|z + 3 - 4i| \leq |z| + 5$.

   (b) **Solution.**
   (We dis-prove the statement ($\star$) by obtaining a contradiction from it.)
   Suppose it were true that there existed some $z \in \mathbb{C}$ such that $|z + 3 - 4i| > |z| + 5$.
   Note that $|z| + 5 \geq 0$.
   Then $|z|^2 + 10|z| + 25 = (|z| + 5)^2 < |z + 3 - 4i|^2 = (z + 3 - 4i)(\bar{z} + 3 + 4i) = |z|^2 + (3 + 4i)z + (3 - 4i)\bar{z} + 25 = |z|^2 + 2\mathsf{Re}((3 + 4i)z) + 25$.
   Therefore $10|z| < 2\mathsf{Re}((3 + 4i)z) \leq 2|(3 + 4i)z| = 2|3 + 4i||z| = 10|z|$. Contradiction arises.
   Hence it is false that there exists some $z \in \mathbb{C}$ such that $|z + 3 - 4i| > |z| + 5$.

   Remark. We may simply quote the Triangle Inequality in the argument:

   Suppose it were true that there existed some $z \in \mathbb{R}$ such that $|z + 3 - 4i| > |z| + 5$.
   By Triangle Inequality, we have $|z + 3 - 4i| \leq |z| + |3 - 4i| = |z| + 5$.
   Then $|z + 3 - 4i| \leq |z| + 5 < |z + 3 - 4i|$. Contradiction arises.
   Hence it is false that there exists some $z \in \mathbb{C}$ such that $|z + 3 - 4i| > |z| + 5$.

   Alternative argument:

   The negation of the statement ($\star$) is:
   $\sim$($\star$): For any $z \in \mathbb{C}$, $|z + 3 - 4i| \leq |z| + 5$.
   We verify the statement $\sim$($\star$):
   Pick any $z \in \mathbb{C}$. We have $|z+3-4i|^2 = \cdots = |z|^2+2\mathsf{Re}((3+4i)z)+25 \leq |z|^2+2|(3+4i)z|+25 = \cdots = (|z|+5)^2$.
   Then $|z + 3 - 4i| \leq |z| + 5$.

   Another alternative argument:

   The negation of the statement ($\star$) is:
   $\sim$($\star$): For any $z \in \mathbb{C}$, $|z + 3 - 4i| \leq |z| + 5$.
   Pick any $z \in \mathbb{C}$. Suppose it were true that $|z + 3 - 4i| > |z| + 5$ for this z.
   Note that $|z| + 5 \geq 0$. Then $|z|^2 + 10|z| + 25 = (|z| + 5)^2 < |z + 3 - 4i|^2 = \cdots = |z|^2 + 2\mathsf{Re}((3 + 4i)z) + 25$.
   Then $10|z| < 2\mathsf{Re}((3 + 4i)z) \leq 2|(3 + 4i)z| = \cdots = 10|z|$.
   Contradiction arises.
   Hence $|z + 3 - 4i| \leq |z| + 5$.

2. (a) ——

   (b) **Solution.**
   Method (A).
   Denote by N the statement below:
   N: *There exists some $t \in \mathbb{R}$ such that (for any $s \in \mathbb{C}$, $|s| \leq t$).*
   The negation of N reads:
   $\sim N$: For any $t \in \mathbb{R}$, there exists some $s \in \mathbb{C}$ such that $|s| > t$.
   We verify $\sim N$:
   - Pick any $t \in \mathbb{R}$.
     Take $s = |t| + 1$. By definition, $s \in \mathbb{C}$.
     Note that s is a positive real number. Then $|s| = ||t| + 1| = |t| + 1 > |t| \geq t$.

   Method (B).
   (Denote by N the statement below:
   N: There exists some $t \in \mathbb{R}$ such that (for any $s \in \mathbb{C}$, $|s| \leq t$).
   We dis-prove the statement N by obtaining a contradiction from it.)
   Suppose it were true that there existed some $t \in \mathbb{R}$ such that (for any $s \in \mathbb{C}$, $|s| \leq t$).
   For such a real number t, the statement 'for any $s \in \mathbb{C}$, $|s| \leq t$' would be true.
   Note that $|t| + 1$ is a complex number.
   Then $||t| + 1| \leq t$.
   Since $|t| + 1$ is a non-negative real number, we have $||t| + 1| = |t| + 1$.
   Then we have $|t| + 1 \leq t \leq |t|$. Therefore $1 \leq 0$.
   Contradiction arises.

3. ——

4. ——

5. ——

6. (a) **Answer.**
   Let $c, c' \in I$. Suppose $f(c) = g(c)$ and $f(c') = g(c')$. Then $c = c'$.

   (b) **Solution.**
   Pick any $c, c' \in I$. Suppose $f(c) = g(c)$ and $f(c') = g(c')$. We verify that $c = c'$ by the proof-by-contradiction method:

   - Suppose it were true that $c \neq c'$.
     Without loss of generality, assume $c < c'$.
     Since f is strictly increasing on I, we would have $f(c) < f(c')$.
     Since g is strictly decreasing on I we would have $g(c) > g(c')$.
     Recall that $f(c) = g(c)$ and $f(c') = g(c')$.
     Then $f(c) < f(c') = g(c') < g(c) = f(c)$. Therefore $f(c) < f(c)$. Contradiction arises.

7. ——