MATH1050 Proof-writing Exercise 8

Advice.

- Study the Handouts *Statements with several quantifiers*, *Dis-proofs* (with emphasis on 'wholesale refutation') before answering the questions.
- Besides the handouts mentioned above, Question (7a) of Exercise 7 is also suggestive on what it takes to give a correct argument with wholesale refutation.
- Besides the handouts mentioned above, Question (5a) of Exercise 7 is also suggestive on what it takes to give a correct argument for a uniqueness statement.

1. Consider the statement ($\star$) below.

   ($\star$) *There exists some $z \in \mathbb{C}$ such that $|z + 3 - 4i| > |z| + 5$.*

   (a) Write down the negation of the statement ($\star$).
   (b) Dis-prove the statement ($\star$) by proving the negation of the statement ($\star$), or by obtaining a contradiction from the statement ($\star$) as an assumption.

2. Dis-prove the statements below:

   (a) *There exists some $r \in \mathbb{R}$ such that $r < r^5 \leq r^3$.*
   (b) *There exists some $t \in \mathbb{R}$ such that (for any $s \in \mathbb{C}$, $|s| \leq t$).*

3. (a) Dis-prove the statement ($\star$):

   ($\star$) *There exist some positive real numbers x, y such that $(x + y)^2 \leq x^2 + y^2$.*

   (b) Hence dis-prove the statement ($\star\star$):

   ($\star\star$) *There exist some positive real numbers u, v such that $\sqrt{u} + \sqrt{v} \leq \sqrt{u + v}$.*

4. Dis-prove the statement ($\star$):

   ($\star$) *There exists some $k \in \mathbb{N} \backslash \{0, 1\}$ such that for any positive integer n, the number $k^{1/n}$ is an integer.*

   Remark. You will probably need the Well-ordering Principle for Integers.

5. Let $A = \{x \in \mathbb{R} : x = a + b\sqrt{2} \text{ for some } a, b \in \mathbb{Q}\}$, $B = \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right)$ and $C = A \cap B$.

   For each of the statements below, decide whether it is true or false. If it is true, give a proof. If it is false, give a dis-proof.

   (a) C has a least element.
   (b) C has a greatest element.

6. Let I be an interval in $\mathbb{R}$, and $f, g : I \longrightarrow \mathbb{R}$ be functions. Suppose f is strictly increasing on I and g is strictly decreasing on I.
   Consider the statement ($*$) below.

   ($*$) *There is at most one $c \in I$ such that $f(c) = g(c)$.*

   (a) Fill in the blanks below in such a way that the resultant statement ($*'$) is a re-formulation of ($*$):

   ($*'$) *Let $c, c' \in I$. Suppose ________ and ________. Then ____________.*

   (b) Prove the statement ($*$), with reference to the definition of *strict monotonicity*.
   Remark. Refer to *Proof-Writing Exercise 2* for the definition of *strict monotonicity*.

7. We introduce/recall the notion of *linear independence* (for vectors):

   Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k \in \mathbb{R}^n$. Suppose $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k$ are pairwise distinct.
   Then $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k$ are **linearly independent** *if the statement* (LI) *holds:*

   (LI) *For any $a_1, a_2, \cdots, a_k \in \mathbb{R}$, if $a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k = \mathbf{0}$ then $a_1 = a_2 = \cdots = a_k = 0$.*

   Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in \mathbb{R}^n$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are linearly independent.
   Prove the statement ($\star$), with reference to the definition of *linearly independence*:

   ($\star$) For any $\mathbf{v} \in \mathbb{R}^n$, there are at most one $c_1 \in \mathbb{R}$, at most one $c_2 \in \mathbb{R}$, ..., and at most one $c_k \in \mathbb{R}$ such that $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$.