MATH1050 Proof-writing Exercise 7

Advice.

- Study the Handouts *Statements with several quantifiers*, *Dis-proofs* (with emphasis on 'dis-proofs by counter-example') before answering the questions.
- Besides the handouts mentioned above, Question (6a) of Exercise 7 is also suggestive on what it takes to give a correct argument with dis-proof by counter-example.

1. Consider the statement ($\star$) below.

   ($\star$) *Let $x, y, z \in \mathbb{N}$. Suppose $x + y$, $y + z$ are divisible by 3. Then $x + z$ is divisible by 3.*

   (a) Write down the negation of the statement ($\star$).

   (b) Dis-prove the statement ($\star$).

2. Dis-prove the statements below:

   (a) *Suppose $x, y \in \mathbb{N}$. Then $\sqrt{x^2 + y^2} \in \mathbb{N}$.*

   (b) *For any $s, t \in \mathbb{R}$, if both of $s + t$, st are rational, then at least one of s, t is rational.*

   (c) *Let n be a positive integer, and ζ be a complex number. Suppose ζ is an n^2-th root of unity. Then ζ^2 is an n-th root of unity.*

3. Dis-prove the statements below:

   (a) *Suppose A, B, C be sets. Then $A \backslash (C \backslash B) \subset A \cap B$.*

   (b) *Suppose A, B, C be non-empty sets. Then $B \backslash A \subset (C \backslash A) \backslash (C \backslash B)$.*

4. Dis-prove the statements below:

   (a) *Let I be an open interval, and $f : I \longrightarrow \mathbb{R}$ be a function. Suppose f is differentiable on I, and f is strictly increasing on I. Then $f'(x) > 0$ for any $x \in I$.*

   (b) *Let I be an open interval, and $f : I \longrightarrow \mathbb{R}$ be a function. Suppose f is differentiable on I, and $f'(x) \geq 0$ for any $x \in I$. Then f is strictly increasing on I.*

   (c) *Let I, J be open intervals , and $f : I \cup J \longrightarrow \mathbb{R}$ be a function. Suppose f is differentiable at every point of $I \cup J$, and $f'(x) = 0$ for any $x \in I \cup J$. Then f is constant on $I \cup J$.*

5. We introduce/recall the definition for the notion of *non-singularity* for square matrices with real entries:

   Let A be an $(m \times m)$-square matrix with real entries. The matrix A is said to be **non-singular** *if the statement* (NS) *holds:*

   (NS) *For any $\mathbf{x} \in \mathbb{R}^m$, if $A\mathbf{x} = \mathbf{0}$ then $\mathbf{x} = \mathbf{0}$.*

   Dis-prove the statements below.

   (a) *Let A, B be non-zero (2×2)-square matrices with real entries. Suppose A, B are non-singular and $A + B$ is not the zero matrix. Then $A + B$ is non-singular.*

   (b) *Let A, B be non-zero (2×2)-square matrices with real entries. Suppose A, B are non-singular and $A + B$ is not the zero matrix. Then $A + B$ is singular.*

   (c) *Let A, B be non-zero (2×2)-square matrices with real entries. Suppose A, B are singular and $A + B$ is not the zero matrix. Then $A + B$ is singular.*

   (d) *Let A, B be non-zero (2×2)-square matrices with real entries. Suppose A, B are singular and $A + B$ is not the zero matrix. Then $A + B$ is non-singular.*