MATH1050 Proof-writing Exercise 5 (Answers and selected solution)

1. **Solution.**

Acceptable argument (A).

Let r be a real number greater than 1. Denote by $P(n)$ the proposition below:

Suppose $a_1, a_2, \cdots, a_n$ are positive real numbers. Then $\log_r\left(\prod_{j=1}^{n} a_j\right) = \sum_{j=1}^{n} \log_r(a_j)$.

- Suppose a, b are positive real numbers. Then $\log_r(ab) = \log_r(a) + \log_r(b)$ by ($\sharp$).
  It follows that $P(2)$ is true.
- Let $k \in \mathbb{N}\backslash\{0,1\}$. Suppose $P(k)$ is true. We verify that $P(k+1)$ is true:
  - Suppose $a_1, a_2, \cdots, a_k, a_{k+1}$ are positive real numbers.
    Since $a_1, a_2, \cdots, a_k$ are positive real numbers, $a_1 a_2 \cdots a_k$ is a positive real number.
    Then

$$\begin{aligned}
\log_r\left(\prod_{j=1}^{k+1} a_j\right) &= \log_r\left(\left(\prod_{j=1}^{k} a_j\right) \cdot a_{k+1}\right) \\
&= \log_r\left(\prod_{j=1}^{k} a_j\right) + \log_r(a_{k+1}) \qquad \text{(by } (\sharp)) \\
&= \sum_{j=1}^{k} \log_r(a_j) + \log_r(a_{k+1}) \qquad \text{(by } P(k)) \\
&= \sum_{j=1}^{k+1} \log_r(a_j)
\end{aligned}$$

    Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any positive integer $n \in \mathbb{N}\backslash\{0,1\}$.

Acceptable argument (B), but not preferrable.

Let r be a real number greater than 1. Let $\{a_j\}_{j=1}^{\infty}$ be an infinite sequence of positive real numbers. Denote by $S(n)$ the proposition below:

$$\log_r\left(\prod_{j=1}^{n} a_j\right) = \sum_{j=1}^{n} \log_r(a_j).$$

- We have $\log_r(a_1 a_2) = \log_r(a_1) + \log_r(a_2)$ by ($\sharp$).
  Then $S(2)$ is true.
- Let $k \in \mathbb{N}\backslash\{0,1\}$. Suppose $S(k)$ is true. We verify that $S(k+1)$ is true:
  - Since $a_1, a_2, \cdots, a_k$ are positive real numbers, $a_1 a_2 \cdots a_k$ is a positive real number.
    Then

$$\begin{aligned}
\log_r\left(\prod_{j=1}^{k+1} a_j\right) &= \log_r\left(\left(\prod_{j=1}^{k} a_j\right) \cdot a_{k+1}\right) \\
&= \log_r\left(\prod_{j=1}^{k} a_j\right) + \log_r(a_{k+1}) \qquad \text{(by } (\sharp)) \\
&= \sum_{j=1}^{k} \log_r(a_j) + \log_r(a_{k+1}) \qquad \text{(by } S(k)) \\
&= \sum_{j=1}^{k+1} \log_r(a_j)
\end{aligned}$$

Hence $S(k+1)$ is true.

By the Principle of Mathematical Induction, $S(n)$ is true for any positive integer $n \in \mathbb{N} \backslash \{0, 1\}$.

2. ——

3. (a) **Solution.**

Let A, B be $(m \times m)$-square matrices with real entries. Suppose A, B are non-singular.

We verify that AB is non-singular:

Pick any $\mathbf{x} \in \mathbb{R}^m$. Suppose $AB\mathbf{x} = \mathbf{0}$.
We have $A(B\mathbf{x}) = \mathbf{0}$. Then, since A is non-singular, $B\mathbf{x} = \mathbf{0}$.
Then, since B is non-singular, we have $\mathbf{x} = \mathbf{0}$.
It follows that AB is non-singular.

(b) i. **Answer.**

Let n be an integer greater than 1. Let $A_1, A_2, \cdots, A_n$ be $(m \times m)$-square matrices. Suppose $A_1, A_2, \cdots, A_n$ are non-singular. Then $A_1 A_2 \cdots A_n$ is non-singular.

ii. **Solution.**

Denote by $P(n)$ the proposition below:

Let $A_1, A_2, \cdots, A_n$ be $(m \times m)$-square matrices. Suppose $A_1, A_2, \cdots, A_n$ are non-singular. Then $A_1 A_2 \cdots A_n$ is non-singular.

- $P(2)$ is true by the result of part (a).
- Let k be an integer greater than 1. Suppose $P(k)$ is true.
  We verify that $P(k+1)$ is true:

  Let $A_1, A_2, \cdots, A_k, A_{k+1}$ be $(m \times m)$-square matrix.
  Suppose $A_1, A_2, \cdots, A_k, A_{k+1}$ is non-singular. Write $C = A_1 A_2 \cdots A_k$.
  By $P(k)$, since $A_1, A_2, \cdots, A_k$ are non-singular, the product C is non-singular.
  Note that $A_1 A_2 \cdots A_k A_{k+1} = C A_{k+1}$.
  Then, by the result of part (a), $A_1 A_2 \cdots A_k A_{k+1}$ is non-singular.

By the Principle of Mathematical Induction, $P(n)$ is true for any integer greater than 1.

Alternative 'inductive step'.

- Let k be an integer greater than 1. Suppose $P(k)$ is true.
  We verify that $P(k+1)$ is true:

  Let $A_1, A_2, \cdots, A_k, A_{k+1}$ be $(m \times m)$-square matrix.
  Suppose $A_1, A_2, \cdots, A_k, A_{k+1}$ is non-singular.
  By the result in part (a), since A_k, A_{k+1} are non-singular, $A_k A_{k+1}$ is non-singular.
  $A_1, A_2, \cdots, A_{k-1}, A_k A_{k+1}$ are non-singular. Then by $P(k)$, the product $A_1 A_2 \cdots A_{k-1} A_k A_{k+1}$ is non-singular.

4. ——