MATH1050 Proof-writing Exercise 5

Advice.

- Study the Handout *Mathematical induction* before answering the questions.
- Besides the handout mentioned above, Question (7a) of Exercise 4 is also suggestive on what it takes to give a correct argument with mathematical induction.

1. In this question, take for granted the validity of the statement below:

   ($\sharp$) *For any $t, u, v \in \mathbb{R}$, if $t > 0$, $u > 0$, $v > 0$ and $t \neq 1$ then $\log_t(uv) = \log_t(u) + \log_t(v)$.*

   Apply mathematical induction to prove the statement below:

   Let $n \in \mathbb{N}\backslash\{0, 1\}$, and r be a real number greater than 1. Suppose $a_1, a_2, \cdots, a_n$ are positive real numbers. Then $\log_r\left(\prod_{j=1}^{n} a_j\right) = \sum_{j=1}^{n} \log_r(a_j)$.

2. (a) Verify the statements below:

   i. *Let $a, b \in \mathbb{R}$. Suppose $0 \leq a < b$. Then $\dfrac{a}{1+a} \leq \dfrac{b}{1+b}$.*

   ii. *Let $a, b \in \mathbb{R}$. Suppose a, b are non-negative. Then $\dfrac{a+b}{1+a+b} \leq \dfrac{a}{1+a} + \dfrac{b}{1+b}$.*

   (b) Apply mathematical induction to prove the statement below:

   Let $n \in \mathbb{N}\backslash\{0, 1\}$. Suppose $x_1, x_2, \cdots, x_n$ are non-negative real numbers. Then

   $$\frac{x_1 + x_2 + \cdots + x_n}{1 + x_1 + x_2 + \cdots + x_n} \leq \sum_{j=1}^{n} \frac{x_j}{1 + x_j}.$$

3. We introduce/recall the definition for the notion of *non-singularity* for square matrices with real entries:

   Let A be an $(m \times m)$-square matrix with real entries. The matrix A is said to be **non-singular** *if the statement* (NS) *holds:*

   (NS) *For any $\mathbf{x} \in \mathbb{R}^m$, if $A\mathbf{x} = \mathbf{0}$ then $\mathbf{x} = \mathbf{0}$.*

   (a) Prove the statement below, with reference to the definition:

   Let A, B be $(m \times m)$-square matrices with real entries. Suppose A, B are non-singular. Then AB is non-singular.

   (b) Consider the statement ($*$):

   ($*$) *The product of any two or more non-singular $(m \times m)$-square matrices with real entries is non-singular.*

   i. Fill in the blanks below in such a way that the resultant statement ($*'$) is a re-formulation of ($*$):

   ($*'$) *Let n be an integer greater than 1. Let* ____________ *be $(m \times m)$-square matrices with real entries. Suppose* ____________________. *Then* ____________________.

   ii. Apply mathematical induction to prove the statement ($*$).

   Remark. You may take the result in part (a) for granted.

4. Consider the statement ($\star$):

   ($\star$) *Every integer greater than 1 is a prime number or a product of at least two prime numbers.*

   (a) Fill in the blanks below in such a way that the resultant statement ($\star'$) is a re-formulation of ($\star$):

   ($\star'$) *Let n* ____________________. *For any integer m between* __ *and n, the integer m is a prime number or* ______________________________.

   (b) Apply mathematical induction to justify the statement ($\star$).

   Remark. You need to recall the definitions for *divisibility* and for *prime numbers*.