MATH1050 Proof-writing Exercise 4 (Answers and selected solution)

1. (a) ——

   (b) ——

   (c) **Solution.**

   Let ζ be a complex number. Suppose that $|\zeta| \leq \varepsilon$ for any positive real number ε. Further suppose it were true that $\zeta \neq 0$.

   Since $\zeta \neq 0$, we would have $|\zeta| \neq 0$. Then $|\zeta| > 0$.

   Define $\varepsilon = \dfrac{|\zeta|}{2}$. Since $|\zeta| > 0$ and $\dfrac{1}{2} > 0$, we would have $\varepsilon > 0$.

   Then by assumption, $|\zeta| \leq \varepsilon = \dfrac{|\zeta|}{2}$.

   Therefore $\dfrac{|\zeta|}{2} = |\zeta| - \dfrac{|\zeta|}{2} \leq 0$. Hence $|\zeta| = 2 \cdot \dfrac{|\zeta|}{2} \leq 0$ (because $2 > 0$).

   Now we have $0 < |\zeta| \leq 0$. Contradiction arises.

   Hence $\zeta = 0$ in the first place.

2. ——

3. **Solution.**

   Let x be a positive real number, r be a positive rational number, and n be an integer greater than 1. Suppose x is an irrational number.

   Further suppose it were true that $\sqrt[n]{x+r}$ was a rational number.

   Write $y = \sqrt[n]{x+r}$. Note that $y^n - r = x$.

   Since y was a rational number, y^n would be a rational number. Moreover, since r is a rational number, $y^n - r$ would be a rational number.

   Therefore x would be a rational number. But by assumption x is an irrational number.

   Contradiction arises.

   Hence the assumption that $\sqrt[n]{x+r}$ was rational is false. So $\sqrt[n]{x+r}$ is an irrational number in the first place.

4. ——

5. ——

6. ——