MATH1050 Proof-writing Exercise 3 (Answers and selected solution)

1. (a) ——

   (b) ——

   (c) ——

   (d) **Solution.**

   Suppose s, t are Gaussian integers. By definition, $\mathsf{Re}(s), \mathsf{Im}(s), \mathsf{Re}(t), \mathsf{Im}(t)$ are integers.

   - Note that $\bar{s} = \mathsf{Re}(s) - i\mathsf{Im}(s) = \mathsf{Re}(s) + i(-\mathsf{Im}(s))$.
     Note that $\mathsf{Re}(s), -\mathsf{Im}(s)$ are integers.
     Then, by definition, $\bar{s}$ is a Gaussian integer.
   - Note that $s + t = (\mathsf{Re}(s) + \mathsf{Re}(t)) + i(\mathsf{Im}(s) + \mathsf{Im}(t))$.
     Then $\mathsf{Re}(s + t) = \mathsf{Re}(s) + \mathsf{Re}(t)$. Also, $\mathsf{Im}(s + t) = \mathsf{Im}(s) + \mathsf{Im}(t)$.
     Since $\mathsf{Re}(s), \mathsf{Re}(t), \mathsf{Im}(s), \mathsf{Im}(t)$ are integers, $\mathsf{Re}(s + t)$ and $\mathsf{Im}(s + t)$ are integers.
     Hence $s + t$ is a Gaussian integer.
   - Note that $st = (\mathsf{Re}(s)\mathsf{Re}(t) - \mathsf{Im}(s)\mathsf{Im}(t)) + i(\mathsf{Re}(s)\mathsf{Im}(t) + \mathsf{Im}(s)\mathsf{Re}(t))$.
     Then $\mathsf{Re}(st) = \mathsf{Re}(s)\mathsf{Re}(t) - \mathsf{Im}(s)\mathsf{Im}(t)$. Also, $\mathsf{Im}(st) = \mathsf{Re}(s)\mathsf{Im}(t) + \mathsf{Im}(s)\mathsf{Re}(t)$.
     Since $\mathsf{Re}(s), \mathsf{Re}(t), \mathsf{Im}(s), \mathsf{Im}(t)$ are integers, $\mathsf{Re}(st)$ and $\mathsf{Im}(st)$ are integers.
     Hence st is a Gaussian integer.

2. (a) ——

   (b) ——

   (c) ——

   (d) **Solution.**

   Let $u, v, w \in \mathbb{G}$. Suppose u is $\mathbb{G}$-divisible by v and v is $\mathbb{G}$-divisible by w.
   Since u is $\mathbb{G}$-divisible by v, there exists some $s \in \mathbb{G}$ such that $u = sv$.
   Since v is $\mathbb{G}$-divisible by w, there exists some $t \in \mathbb{G}$ such that $v = tw$.
   Then $u = stw$. Since $s, t \in \mathbb{G}$, we have $st \in \mathbb{G}$ by the results in Question (1d) .
   Hence u is $\mathbb{G}$-divisible by w.