# MATH1050 Proof-writing Exercise 2 (Answers and selected solution)

1. (a) **Solution.**

   Let $a, b$ be real numbers, and $h : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $h(x) = x^3 - 3a^2x + b$ for any $x \in \mathbb{R}$. Suppose $a > 0$.

   Pick any $s, t \in [a, +\infty)$. Suppose $s < t$.

   Note that

   $$\begin{aligned} h(t) - h(s) &= (t^3 - 3a^2t + b) - (s^3 - 3a^2s + b) \\ &= (t^3 - s^3) - 3a^2(t - s) \\ &= (t - s)(t^2 + st + s^2 - 3a^2) \end{aligned}$$

   Since $s < t$, we have $t - s > 0$.

   Since $0 < a \leq s < t$, we have $0 < a^2 \leq s^2 < st < t^2$.

   Then $t^2 + st + s^2 > 3s^2 \geq 3a^2$.

   Therefore $t^2 + st + s^2 - 3a^2 > 0$.

   Hence $h(t) - h(s) = (t - s)(t^2 + st + s^2 - 3a^2) > 0$.

   Then $h(s) < h(t)$.

   It follows that $h$ is strictly increasing on the interval $[a, +\infty)$.

   (b) ——

2. ——

3. (a) **Solution.**

   Let $f$ be a real-valued function of one real variable defined on some open interval $I$ in $\mathbb{R}$.

   Suppose $f$ satisfies all the conditions:

   (D) $f$ is differentiable on $I$. (P) $f'(x) > 0$ for any $x \in I$.

   Pick any $s, t \in I$. Suppose $s < t$.

   (Since $I$ is an open interval, the interval $[s, t]$ lies entirely inside $I$. By Condition (D), the function $f$ is continuous on $[s, t]$ and it is differentiable on $(s, t)$.)

   By the Mean-Value Theorem, there exists some $\zeta \in (s, t)$ such that $f(t) - f(s) = (t - s)f'(\zeta)$.

   By Condition (P), $f'(\zeta) > 0$.

   Since $s < t$, we have $t - s > 0$.

   Then $f(t) - f(s) = (t - s)f'(\zeta) > 0$.

   Therefore $f(s) < f(t)$.

   It follows that $f$ is strictly increasing on $I$.

   (b) ——