MATH1050 Proof-writing Exercise 2

Advice.

- Adhere to the definitions (introduced) in the questions, and be patient.
- Before writing anything at all, identify the assumption and the conclusion in the statement to be proved. After writing down the assumption which will be used throughout an argument, think hard (with reference to definition) what the desired conclusion means and what it takes to arrive at it from the assumption that you have written down.
- Make good use of the statement below, concerned with inequalities for real numbers:

  Let u, v be real numbers. The inequality $u < v$ holds iff the inequality $v - u > 0$ holds.

1. We introduce/recall the definitions on *strict monotonicity* for real-valued functions of one real variable:

   Let I be an interval, and $h : D \longrightarrow \mathbb{R}$ be a real-valued function of one real variable with domain D which contains I as a subset entirely.

   - *h is said to be* **strictly increasing** *on I if the statement* (StrIncr) *holds:*

     (StrIncr) *For any $s, t \in I$, if $s < t$ then $h(s) < h(t)$.*
   - *h is said to be* **strictly decreasing** *on I if the statement* (StrDecr) *holds:*

     (StrDecr) *For any $s, t \in I$, if $s < t$ then $h(s) > h(t)$.*

   Prove the statements below, with direct reference to the definitions on strict monotonicity:

   (a) *Let a, b be real numbers, and $h : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $h(x) = x^3 - 3a^2x + b$ for any $x \in \mathbb{R}$. Suppose $a > 0$. Then h is strictly increasing on the interval $[a, +\infty)$.*

   (b) *Let a be a real number, and $h : (0, +\infty) \longrightarrow \mathbb{R}$ be the real-valued function of one real variable defined by $h(x) = x + \dfrac{a^2}{x}$ for any $x \in (0, +\infty)$. Suppose $a > 0$. Then h is strictly decreasing on $(0, a]$.*

2. We introduce/recall the definitions on *strict convexity* for real-valued functions of one real variable:

   Let I be an interval, and $h : D \longrightarrow \mathbb{R}$ be a real-valued function of one real variable with domain D which contains I as a subset entirely.

   h is said to be **strictly convex** *on I if the statement* (StrConv) *holds:*

   (StrConv) *For any $s, t \in I$, for any $\theta \in (0, 1)$, if $s < t$ then $h((1 - \theta)s + \theta t) < (1 - \theta)h(s) + \theta h(t)$.*

   (We can define 'strict concavity' in an analogous manner.)

   Prove the statements below, with direct reference to the definitions on strict convexity:

   (a) *Let a, b be real numbers, and $h : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $h(x) = x^2 + ax + b$ for any $x \in \mathbb{R}$. The function h is strictly convex on $\mathbb{R}$.*

   (b) Let $h : (0, +\infty) \longrightarrow \mathbb{R}$ be the function defined by $h(x) = \dfrac{1}{x}$ for any $x \in (0, +\infty)$. The function h is strictly convex on $(0, +\infty)$.

3. Take for granted the validity of the statement below, known as **Mean-Value Theorem**:

   Let $a, b \in \mathbb{R}$, with $a < b$, and f be a real-valued function of one real variable defined on $[a, b]$.
   Suppose f is continuous on $[a, b]$ and f is differentiable on (a, b).
   Then there exists some $\zeta \in (a, b)$ such that $f(b) - f(a) = (b - a)f'(\zeta)$.

   (a) Apply the Mean-Value Theorem to deduce the statement (SI) below, (which relates strict monotonicity to the sign of the first derivative):

   (SI) *Let f be a real-valued function of one real variable defined on some open interval I in $\mathbb{R}$.*
   Suppose f satisfies all the conditions:

   (D) *f is differentiable on I.* (P) *$f'(x) > 0$ for any $x \in I$.*

   Then f is strictly increasing on I.

   (b) Apply the Mean-Value Theorem and the statement (SI), to prove the statement (SV) (which relates strict convexity/concavity to the sign of the second derivative):

   (SV) *Let f be a real-valued function of one real variable defined on some open interval I in $\mathbb{R}$.*
   Suppose f satisfies all the conditions:

   (D2) *f is twice differentiable on I.* (P2) *$f''(x) > 0$ for any $x \in I$.*

   Then f is strictly convex on I.