MATH1050 Workshop on Proof-writing 1 (Solution)

1. Let $x, n \in \mathbb{Z}$. Suppose x is divisible by n.
   Let $y \in \mathbb{Z}$.
   By assumption, there exists some $t \in \mathbb{Z}$ such that $x = tn$.

$$\begin{aligned}
& (5x+y)^7 + (5x-y)^7 \\
= \ & [(5x+y)+(5x-y)][(5x+y)^6 - (5x+y)^5(5x-y) + (5x+y)^4(5x-y)^2 - (5x+y)^3(5x-y)^3 \\
& +(5x+y)^2(5x-y)^4 - (5x+y)(5x-y)^5 + (5x-y)^6] \\
= \ & 10x[(5x+y)^6 - (5x+y)^5(5x-y) + (5x+y)^4(5x-y)^2 - (5x+y)^3(5x-y)^3 \\
& +(5x+y)^2(5x-y)^4 - (5x+y)(5x-y)^5 + (5x-y)^6] \\
= \ & 10n \cdot t[(5x+y)^6 - (5x+y)^5(5x-y) + (5x+y)^4(5x-y)^2 - (5x+y)^3(5x-y)^3 \\
& +(5x+y)^2(5x-y)^4 - (5x+y)(5x-y)^5 + (5x-y)^6]
\end{aligned}$$

   Since t, x, y are integers, $t[(5x+y)^6 - (5x+y)^5(5x-y) + (5x+y)^4(5x-y)^2 - (5x+y)^3(5x-y)^3 + (5x+y)^2(5x-y)^4 - (5x+y)(5x-y)^5 + (5x-y)^6]$ is also an integer.
   Then, by definition, $(5x+y)^7 + (5x-y)^7$ is divisible by $10n$.

2. Let x, y be rational numbers. Suppose $y \neq 0$.
   Since x is a rational number, there exist some $m, n \in \mathbb{Z}$ such that $n \neq 0$ and $m = nx$.
   Since y is a rational number, there exist some $p, q \in \mathbb{Z}$ such that $q \neq 0$ and $p = qy$.
   Since $y \neq 0$, we have $p \neq 0$. Reacll $n \neq 0$. Then $pn \neq 0$.
   We also have $mq \cdot y = m \cdot qy = nx \cdot p = np \cdot x = np \cdot \dfrac{x}{y} \cdot y$.
   Since $y \neq 0$, we have $mq = np \cdot \dfrac{x}{y}$.
   It follows that $\dfrac{x}{y}$ is a rational number.

3. Let p, q be prime numbers. Suppose p, q are positive and $p \neq q$.
   Suppose it were true that p was divisible by q.
   Since p is a prime number, p is divisible by $1, -1, p, -p$ only.
   Then, since p was divisible by q, $q = 1$ or $q = -1$ or $q = p$ or $q = -p$.
   Since q is a prime number, $q \neq 1$ and $q \neq -1$.
   By assumption, $q \neq p$.
   Since p is positive, $-p$ is negative. Then by assumption, $q \neq -p$.
   Contradiction arises.
   It follows that p is not divisible by q in the first place.

4. Let a be a real number, and $h : (0, +\infty) \longrightarrow \mathbb{R}$ be the real-valued function of one real variable defined by $h(x) = x + \dfrac{a^2}{x}$ for any $x \in (0, +\infty)$. Suppose $a > 0$.
   Pick any $s, t \in (0, a]$. Suppose $s < t$.
   Note that

$$h(s) - h(t) = (s + \frac{a^2}{s}) - (t + \frac{a^2}{t}) = (s-t) + a^2(\frac{1}{s} - \frac{1}{t}) = (s-t) + \frac{a^2}{st}(t-s) = \frac{t-s}{st} \cdot (a^2 - st)$$

   Since $0 < s < t$, we have $t - s > 0$ and $st > 0$. Then $\dfrac{t-s}{st} > 0$.
   Since $0 < s < t \leq a$, we have $0 < st < t^2 \leq a^2$. Then $a^2 - st > 0$.
   Therefore $h(s) - h(t) = \dfrac{t-s}{st} \cdot (a^2 - st) > 0$. Then $h(t) < h(s)$.
   It follows that h is strictly decreasing on $(0, a]$.

5. Let a, b be real numbers. Suppose $|a| \leq 1$ and $|b| \leq 1$. Suppose $\sqrt{1-a^2} + \sqrt{1-b^2} > 2\sqrt{1-(a+b)^2/4}$.

Note that $2\sqrt{1 - \dfrac{(a+b)^2}{4}} \geq 0$.

$$\begin{aligned}(1-a^2) + (1-b^2) + 2\sqrt{(1-a^2)(1-b^2)} &= (\sqrt{1-a^2} + \sqrt{1-b^2})^2 \\ &> \left[2\sqrt{1 - \frac{(a+b)^2}{4}}\right]^2 = 4\left[1 - \frac{(a+b)^2}{4}\right] = 4 - (a+b)^2\end{aligned}$$

Therefore (after simplification), we would obtain $\sqrt{(1-a^2)(1-b^2)} > 1 - ab$.

Since $|a| \leq 1$ and $|b| \leq 1$, we have $ab \leq |ab| \leq 1$. Then $1 - ab \geq 0$.

Then $1 - a^2 - b^2 + a^2b^2 = (1-a^2)(1-b^2) > (1-ab)^2 = 1 - 2ab + a^2b^2$.

Therefore (after simplification), we would obtain $a^2 + b^2 - 2ab < 0$.

But $a^2 + b^2 - 2ab = (a-b)^2 \geq 0$ because a, b are real numbers. Contradiction arises.

Hence $\sqrt{1-a^2} + \sqrt{1-b^2} \leq 2\sqrt{1 - \dfrac{(a+b)^2}{4}}$ in the first place.

6. Let $m, n \in \mathbb{N}\backslash\{0\}$. Let x be a positive real number. Suppose $m > n$.

We have $\left(x^m + \dfrac{1}{x^m}\right) - \left(x^n + \dfrac{1}{x^n}\right) = (x^m - x^n) - \dfrac{x^m - x^n}{x^{m+n}} = \dfrac{(x^m - x^n)(x^{m+n} - 1)}{x^{m+n}}$.

Note that $x^{m+n} > 0$. We verify that $(x^m - x^n)(x^{m+n} - 1) \geq 0$.

- (Case 1). Suppose $0 < x < 1$. Then $0 < x^{m-n} < 1$ and $x^n > 0$.
  Therefore $x^m - x^n = (x^{m-n} - 1)x^n < 0$.
  Hence $x^m - x^n < 0$.
  Also $0 < x^{m+n} < 1$. Then $x^{m+n} - 1 < 0$. It follows that $(x^m - x^n)(x^{m+n} - 1) > 0$.
- (Case 2). Suppose $x \geq 1$. Then $x^{m-n} \geq 1$ and $x^n > 0$.
  Therefore $x^m - x^n = (x^{m-n} - 1) \cdot x^n > 0$.
  Hence $x^m - x^n \geq 0$.
  Also $x^{m+n} \geq 1$. Then $x^{m+n} - 1 \geq 0$. It follows that $(x^m - x^n)(x^{m+n} - 1) \geq 0$.

Therefore, in any case, $(x^m - x^n)(x^{m+n} - 1) \geq 0$.

It follows that $\left(x^m + \dfrac{1}{x^m}\right) - \left(x^n + \dfrac{1}{x^n}\right) = \dfrac{(x^m - x^n)(x^{m+n} - 1)}{x^{m+n}} \geq 0$. Hence $x^m + \dfrac{1}{x^m} \geq x^n + \dfrac{1}{x^n}$.

- Suppose $x = 1$. Then $x^m + \dfrac{1}{x^m} = 2 = x^n + \dfrac{1}{x^n}$.
- Suppose $x^m + \dfrac{1}{x^m} = x^n + \dfrac{1}{x^n}$. Then $0 = \dfrac{(x^m - x^n)(x^{m+n} - 1)}{x^{m+n}}$. Therefore $x^m = x^n$ or $x^{m+n} = 1$.
  * (Case 1). Suppose $x^m = x^n$. Then $x^{m-n} = 1$. Since $m > n$, we have $m - n > 0$. Since $x > 0$, we have $x = 1$.
  * (Case 2). Suppose $x^{m+n} = 1$. Note that $m + n > 0$. Since $x > 0$, we have $x = 1$.

  Hence, in any case, we have $x = 1$.