1. Prove the statement ($\star$) below, with reference to the definition for *divisibility*:

   ($\star$) *Let $x, n \in \mathbb{Z}$. Suppose x is divisible by n. Then for any $y \in \mathbb{Z}$, $(5x + y)^7 + (5x - y)^7$ is divisible by $10n$.*

   [Ask. What to write at the beginning of the argument?
   Ask. What is our objective?]

Prove the statement ($\star$) below, with reference to the definition for *divisibility*:

($\star$) *Let* $x, n \in \mathbb{Z}$. *Suppose* x *is divisible by* n. *Then for any* $y \in \mathbb{Z}$, $(5x+y)^7 + (5x-y)^7$ *is divisible by* $10n$.

[Ask. What to write at the beginning of the argument?
Answer. '*Let* $x, n \in \mathbb{Z}$. *Suppose* x *is divisible by* n.'
Ask. What is our objective?
Answer. To deduce '*for any* $y \in \mathbb{Z}$, $(5x+y)^7 + (5x-y)^7$ *is divisible by* $10n$'.
To make life easier, after introducing x, n and the assumption on x, n, we write '*Let* $y \in \mathbb{Z}$'.
We try to deduce (for such a y), it will happen that '$(5x+y)^7 + (5x-y)^7$ *is divisible by* $10n$'.
Now ask. How to deduce '$(5x+y)^7 + (5x-y)^7$ *is divisible by* $10n$'. What does the definition of divisibility suggest we do?]

Prove the statement ($\star$) below, with reference to the definition for ***divisibility***:

($\star$) *Let* $x, n \in \mathbb{Z}$. *Suppose* x *is divisible by* n. *Then for any* $y \in \mathbb{Z}$, $(5x + y)^7 + (5x - y)^7$ *is divisible by* $10n$.

Let $x, n \in \mathbb{Z}$. Suppose x is divisible by n.
Let $y \in \mathbb{Z}$.

[Ask. How to deduce '$(5x + y)^7 + (5x - y)^7$ *is divisible by* $10n$'. What does the definition of divisibility suggest?
Answer. '*There exists some integer* k *such that* $(5x + y)^7 + (5x - y)^7 = 10n \cdot k$.'
So we want to name an appropriate integer k which satisfies $(5x+y)^7+(5x-y)^7 = 10n \cdot k$, and proceed to verify that it is the case indeed.
We need $10n$ to turn up from the expression '$(5x + y)^7 + (5x - y)^7$'. How?
Look at the assumption '*Suppose* x *is divisible by* n'.
Ask. Can we extract 'x' from the '$(5x + y)^7 + (5x - y)^7$'?]

Prove the statement ($\star$) below, with reference to the definition for *divisibility*:

($\star$) *Let* $x, n \in \mathbb{Z}$. *Suppose* x *is divisible by* n. *Then for any* $y \in \mathbb{Z}$, $(5x + y)^7 + (5x - y)^7$ *is divisible by* $10n$.

Let $x, n \in \mathbb{Z}$. Suppose x is divisible by n.
Let $y \in \mathbb{Z}$.

[We want to deduce: '*There exists some integer* k *such that* $(5x + y)^7 + (5x - y)^7 = 10n \cdot k$.'

By assumption, there exists some $t \in \mathbb{Z}$ such that $x = tn$.

Roughwork.
We need $10n$ to turn up from the expression '$(5x + y)^7 + (5x - y)^7$'.

$$\begin{aligned}
& (5x + y)^7 + (5x - y)^7 \\
= \ & [(5x + y) + (5x - y)][(5x + y)^6 - (5x + y)^5(5x - y) + (5x + y)^4(5x - y)^2 - (5x + y)^3(5x - y)^3 \\
& +(5x + y)^2(5x - y)^4 - (5x + y)(5x - y)^5 + (5x - y)^6] \\
= \ & 10x[(5x + y)^6 - (5x + y)^5(5x - y) + (5x + y)^4(5x - y)^2 - (5x + y)^3(5x - y)^3 \\
& +(5x + y)^2(5x - y)^4 - (5x + y)(5x - y)^5 + (5x - y)^6] \\
= \ & 10n \cdot t[(5x + y)^6 - (5x + y)^5(5x - y) + (5x + y)^4(5x - y)^2 - (5x + y)^3(5x - y)^3 \\
& +(5x + y)^2(5x - y)^4 - (5x + y)(5x - y)^5 + (5x - y)^6]
\end{aligned}$$

(Alternative method: Apply Binomial Theorem.)]

Prove the statement ($\star$) below, with reference to the definition for *divisibility*:

($\star$) *Let $x, n \in \mathbb{Z}$. Suppose x is divisible by n. Then for any $y \in \mathbb{Z}$, $(5x+y)^7 + (5x-y)^7$ is divisible by $10n$.*

Let $x, n \in \mathbb{Z}$. Suppose x is divisible by n.
Let $y \in \mathbb{Z}$.

[We want to deduce: '*There exists some integer k such that $(5x+y)^7 + (5x-y)^7 = 10n \cdot k$.*'

By assumption, there exists some $t \in \mathbb{Z}$ such that $x = tn$.

$$\begin{aligned}
& (5x+y)^7 + (5x-y)^7 \\
= \; & [(5x+y)+(5x-y)][(5x+y)^6 - (5x+y)^5(5x-y) + (5x+y)^4(5x-y)^2 - (5x+y)^3(5x-y)^3 \\
& +(5x+y)^2(5x-y)^4 - (5x+y)(5x-y)^5 + (5x-y)^6] \\
= \; & 10x[(5x+y)^6 - (5x+y)^5(5x-y) + (5x+y)^4(5x-y)^2 - (5x+y)^3(5x-y)^3 \\
& +(5x+y)^2(5x-y)^4 - (5x+y)(5x-y)^5 + (5x-y)^6] \\
= \; & 10n \cdot t[(5x+y)^6 - (5x+y)^5(5x-y) + (5x+y)^4(5x-y)^2 - (5x+y)^3(5x-y)^3 \\
& +(5x+y)^2(5x-y)^4 - (5x+y)(5x-y)^5 + (5x-y)^6]
\end{aligned}$$

Since t, x, y are integers, $t[(5x+y)^6 - (5x+y)^5(5x-y) + (5x+y)^4(5x-y)^2 - (5x+y)^3(5x-y)^3 + (5x+y)^2(5x-y)^4 - (5x+y)(5x-y)^5 + (5x-y)^6]$ is also an integer.
Then, by definition, $(5x+y)^7 + (5x-y)^7$ is divisible by $10n$.

2. Prove the statement ($\star$) below, with reference to the definition for *rational numbers*:

($\star$) *Let x, y be rational numbers. Suppose $y \neq 0$. Then $\dfrac{x}{y}$ is a rational number.*

[Ask. What to write at the beginning of the argument?
Ask. What is our objective?]

Prove the statement ($\star$) below, with reference to the definition for rational numbers:

($\star$) *Let x, y be rational numbers. Suppose $y \neq 0$. Then $\dfrac{x}{y}$ is a rational number.*

[Ask. What to write at the beginning of the argument?
Answer. '*Let x, y be rational numbers. Suppose $y \neq 0$.*'
Ask. What is our objective?
Answer. To deduce '$\dfrac{x}{y}$ *is a rational number*'.]

Prove the statement ($\star$) below, with reference to the definition for *rational numbers*:

($\star$) *Let x, y be rational numbers. Suppose $y \neq 0$. Then $\dfrac{x}{y}$ is a rational number.*

Let x, y be rational numbers. Suppose $y \neq 0$.

[Ask. What is our objective?

Answer. To deduce '$\dfrac{x}{y}$ *is a rational number*'.

Ask. What to deduce actually? What does the definition of rational number suggest we should be deducing?]

Prove the statement ($\star$) below, with reference to the definition for *rational numbers*:

($\star$) *Let* x, y *be rational numbers. Suppose* $y \neq 0$. *Then* $\dfrac{x}{y}$ *is a rational number.*

Let x, y be rational numbers. Suppose $y \neq 0$.

[Ask. What is our objective?

Answer. To deduce '$\dfrac{x}{y}$ *is a rational number*'.

Ask. What to deduce actually? What does the definition of rational number suggest we should be deducing?

Answer. '*There exist some* $s, t \in \mathbb{Z}$ *such that* $t \neq 0$ *and* $s = t \cdot \dfrac{x}{y}$.'

We want to name some appropriate s, t for which s is an integer, t is a non-zero integer, and $s = t \cdot \dfrac{x}{y}$.

Ask. How? What does 'x, y *are rational numbers*' suggest according to definition?]

Prove the statement ($\star$) below, with reference to the definition for *rational numbers*:

($\star$) *Let* x, y *be rational numbers. Suppose* $y \neq 0$*. Then* $\frac{x}{y}$ *is a rational number.*

Let x, y be rational numbers. Suppose $y \neq 0$.

[Want to deduce '$\frac{x}{y}$ *is a rational number*'.

We want to name some appropriate s, t for which s is an integer, t is a non-zero integer, and $s = t \cdot \frac{x}{y}$.

Roughwork.
'x, y *are rational numbers*' gives:
There exist some $m, n \in \mathbb{Z}$ such that $n \neq 0$ and $m = nx$.
There exist some $p, q \in \mathbb{Z}$ such that $q \neq 0$ and $p = qy$.

So we expect (†): $\frac{m}{p} = \frac{n}{q} \cdot \frac{x}{y}$.

But we an equality of the form '*Integer* s = *Integer* t *times* $\frac{x}{y}$'.

So we re-write (†): $mq = pn \cdot \frac{x}{y}$.

We are ready to write up the formal argument.]

Prove the statement $(\star)$ below, with reference to the definition for *rational numbers*:

$(\star)$ *Let x, y be rational numbers. Suppose $y \neq 0$. Then $\frac{x}{y}$ is a rational number.*

Let x, y be rational numbers. Suppose $y \neq 0$.

[Want to deduce '$\frac{x}{y}$ *is a rational number*'.

We want to name some appropriate s, t for which s is an integer, t is a non-zero integer, and $s = t \cdot \frac{x}{y}$.]

Since x is a rational number, there exist some $m, n \in \mathbb{Z}$ such that $n \neq 0$ and $m = nx$.
Since y is a rational number, there exist some $p, q \in \mathbb{Z}$ such that $q \neq 0$ and $p = qy$.

[Ask: Is it true that $mq = pn \cdot \frac{x}{y}$? Is it true that $pn \neq 0$?]

Since $y \neq 0$, we have $p \neq 0$. Reacll $n \neq 0$. Then $pn \neq 0$.

We also have $mq \cdot y = m \cdot qy = nx \cdot p = np \cdot x = np \cdot \frac{x}{y} \cdot y$.

Since $y \neq 0$, we have $mq = np \cdot \frac{x}{y}$.

It follows that $\frac{x}{y}$ is a rational number.

3. Prove the statement $(\star)$ below, with reference to the definitions for *divisibility* and *prime number*:

$(\star)$ *Let p, q be prime numbers. Suppose p, q are positive and $p \neq q$. Then p is not divisible by q.*

[Apply proof-by-contradiction method.
Ask. What to write at the beginning of the argument.
Ask. What is our objective?]

Prove the statement ($\star$) below, with reference to the definitions for *divisibility* and *prime number*:

($\star$) *Let p, q be prime numbers. Suppose p, q are positive and $p \neq q$. Then p is not divisible by q.*

[Apply proof-by-contradiction method.
Ask. What to write at the beginning of the argument?
Answer. *Let p, q be prime numbers. Suppose p, q are positive and $p \neq q$.*
Suppose it were true that p was divisible by q.
Ask. What is our objective?
Answer. To obtain a contradiction out of what has been written down as assumption.]

Prove the statement ($\star$) below, with reference to the definitions for *divisibility* and *prime number*:

($\star$) *Let p, q be prime numbers. Suppose p, q are positive and $p \neq q$. Then p is not divisible by q.*

[Apply proof-by-contradiction method.]

Let p, q be prime numbers. Suppose p, q are positive and $p \neq q$.
Suppose it were true that p was divisible by q.

[Ask. What is our objective?
Answer. To obtain a contradiction out of what has been written down as assumption.
Ask. How?
Further ask. What does definition of prime number says about '*p is a prime number*'?]

Prove the statement ($\star$) below, with reference to the definitions for *divisibility* and *prime number*:

($\star$) *Let p, q be prime numbers. Suppose p, q are positive and $p \neq q$. Then p is not divisible by q.*

[Apply proof-by-contradiction method.]

Let p, q be prime numbers. Suppose p, q are positive and $p \neq q$.
Suppose it were true that p was divisible by q.

[Ask. What is our objective?
Answer. To obtain a contradiction out of what has been written down as assumption.
Ask. How? What does definition of prime number says about '*p is a prime number*'?
Answer. '*p is divisible by $1, -1, p, -p$ only.*'
Now ask. Combine this with '*p is divisible by q*'. What does it tell us about q?]

Prove the statement ($\star$) below, with reference to the definitions for *divisibility* and *prime number*:

($\star$) *Let* p, q *be prime numbers. Suppose* p, q *are positive and* $p \neq q$. *Then* p *is not divisible by* q.

[Apply proof-by-contradiction method.]

Let p, q be prime numbers. Suppose p, q are positive and $p \neq q$.
Suppose it were true that p was divisible by q.

[Ask. What is our objective?
Answer. To obtain a contradiction out of what has been written down as assumption.
Ask. How? What does definition of prime number says about '*p is a prime number*'?
Answer. '*p is divisible by* $1, -1, p, -p$ *only.*'
Now ask. Combine this with '*p is divisible by q*'. What does it tell us about q?
Answer. 'q is amongst $1, -1, p, -p$'.
Next ask. Now can we obtain a contradiction, by showing q is not amongst any of them?]

Prove the statement ($\star$) below, with reference to the definitions for *divisibility* and *prime number*:

($\star$) *Let p, q be prime numbers. Suppose p, q are positive and $p \neq q$. Then p is not divisible by q.*

[Apply proof-by-contradiction method.]

Let p, q be prime numbers. Suppose p, q are positive and $p \neq q$.
Suppose it were true that p was divisible by q.
Since p is a prime number, p is divisible by $1, -1, p, -p$ only.
Then, since p was divisible by q, $q = 1$ or $q = -1$ or $q = p$ or $q = -p$.
Since q is a prime number, $q \neq 1$ and $q \neq -1$.
By assumption, $q \neq p$.
Since p is positive, $-p$ is negative. Then by assumption, $q \neq -p$.
Contradiction arises.
It follows that p is not divisible by q in the first place.

4. Prove the statement $(\star)$ below, with reference to the definition of *strict monotonicity*:

$(\star)$ *Let a be a real number, and $h : (0, +\infty) \longrightarrow \mathbb{R}$ be the real-valued function of one real variable defined by $h(x) = x + \dfrac{a^2}{x}$ for any $x \in (0, +\infty)$. Suppose $a > 0$. Then h is strictly decreasing on $(0, a]$.*

Let a be a real number, and $h : (0, +\infty) \longrightarrow \mathbb{R}$ be the real-valued function of one real variable defined by $h(x) = x + \dfrac{a^2}{x}$ for any $x \in (0, +\infty)$. Suppose $a > 0$.

[Ask. What is our objective?
Ask. What is it according to definition?]

Prove the statement ($\star$) below, with reference to the definition of *strict monotonicity*:

($\star$) *Let a be a real number, and $h : (0, +\infty) \longrightarrow \mathbb{R}$ be the real-valued function of one real variable defined by $h(x) = x + \dfrac{a^2}{x}$ for any $x \in (0, +\infty)$. Suppose $a > 0$. Then h is strictly decreasing on $(0, a]$.*

Let a be a real number, and $h : (0, +\infty) \longrightarrow \mathbb{R}$ be the real-valued function of one real variable defined by $h(x) = x + \dfrac{a^2}{x}$ for any $x \in (0, +\infty)$. Suppose $a > 0$.

[Ask. What is our objective?
Answer. We want to deduce '*h is strictly decreasing on* $(0, a]$'.
Ask. What is it according to definition?
Answer. '*For any $s, t \in (0, a]$, if $s < t$ then $h(s) > h(t)$.*'
Ask. So, what to write next? And what to reach accordingly?]

Prove the statement $(\star)$ below, with reference to the definition of *strict monotonicity*:

$(\star)$ *Let a be a real number, and $h : (0, +\infty) \longrightarrow \mathbb{R}$ be the real-valued function of one real variable defined by $h(x) = x + \dfrac{a^2}{x}$ for any $x \in (0, +\infty)$. Suppose $a > 0$. Then h is strictly decreasing on $(0, a]$.*

Let a be a real number, and $h : (0, +\infty) \longrightarrow \mathbb{R}$ be the real-valued function of one real variable defined by $h(x) = x + \dfrac{a^2}{x}$ for any $x \in (0, +\infty)$. Suppose $a > 0$.

[We want to deduce '*For any $s, t \in (0, a]$, if $s < t$ then $h(s) > h(t)$.*'

Pick any $s, t \in (0, a]$. Suppose $s < t$.

[We want to deduce $h(s) > h(t)$.
Recall: For any $u, v \in \mathbb{R}$, $u > v$ iff $u - v > 0$.
It is easier to check whether something is positive by examining its factors.
So, can we re-express $h(s) - h(t)$ as a product of several appropriate factors?]

Prove the statement ($\star$) below, with reference to the definition of *strict monotonicity*:

($\star$) *Let a be a real number, and $h : (0, +\infty) \longrightarrow \mathbb{R}$ be the real-valued function of one real variable defined by $h(x) = x + \dfrac{a^2}{x}$ for any $x \in (0, +\infty)$. Suppose $a > 0$. Then h is strictly decreasing on $(0, a]$.*

Let a be a real number, and $h : (0, +\infty) \longrightarrow \mathbb{R}$ be the real-valued function of one real variable defined by $h(x) = x + \dfrac{a^2}{x}$ for any $x \in (0, +\infty)$. Suppose $a > 0$.

[We want to deduce '*For any $s, t \in (0, a]$, if $s < t$ then $h(s) > h(t)$.*']

Pick any $s, t \in (0, a]$. Suppose $s < t$.

[We want to deduce $h(s) > h(t)$.]

Note that

$$h(s) - h(t) = (s + \frac{a^2}{s}) - (t + \frac{a^2}{t}) = (s - t) + a^2(\frac{1}{s} - \frac{1}{t}) = (s - t) + \frac{a^2}{st}(t - s) = \frac{t - s}{st} \cdot (a^2 - st)$$

[Ask. What can we say about the factors involved?]

Prove the statement ($\star$) below, with reference to the definition of *strict monotonicity*:

($\star$) *Let a be a real number, and $h : (0, +\infty) \longrightarrow \mathbb{R}$ be the real-valued function of one real variable defined by $h(x) = x + \dfrac{a^2}{x}$ for any $x \in (0, +\infty)$. Suppose $a > 0$. Then h is strictly decreasing on $(0, a]$.*

Let a be a real number, and $h : (0, +\infty) \longrightarrow \mathbb{R}$ be the real-valued function of one real variable defined by $h(x) = x + \dfrac{a^2}{x}$ for any $x \in (0, +\infty)$. Suppose $a > 0$.

[We want to deduce '*For any $s, t \in (0, a]$, if $s < t$ then $h(s) > h(t)$.*']

Pick any $s, t \in (0, a]$. Suppose $s < t$.

[We want to deduce $h(s) > h(t)$.]

Note that

$$h(s) - h(t) = (s + \frac{a^2}{s}) - (t + \frac{a^2}{t}) = (s - t) + a^2(\frac{1}{s} - \frac{1}{t}) = (s - t) + \frac{a^2}{st}(t - s) = \frac{t - s}{st} \cdot (a^2 - st)$$

[Ask. What can we say about the factors involved?]

Since $0 < s < t$, we have $t - s > 0$ and $st > 0$. Then $\dfrac{t - s}{st} > 0$.

Since $0 < s < t \leq a$, we have $0 < st < t^2 \leq a^2$. Then $a^2 - st > 0$.

Therefore $h(s) - h(t) = \dfrac{t - s}{st} \cdot (a^2 - st) > 0$. Then $h(t) < h(s)$.

It follows that h is strictly decreasing on $(0, a]$.

5. Prove the statement $(\star)$ below:

$(\star)$ *Let a, b be real numbers. Suppose $|a| \leq 1$ and $|b| \leq 1$. Then* $\sqrt{1-a^2} + \sqrt{1-b^2} \leq 2\sqrt{1-(a+b)^2/4}$.

[Apply proof-by-contradiction method.
Ask. What to write at the beginning of the argument.
Ask. What is our objective?]

Prove the statement ($\star$) below:

($\star$) *Let a, b be real numbers. Suppose $|a| \leq 1$ and $|b| \leq 1$. Then* $\sqrt{1-a^2} + \sqrt{1-b^2} \leq 2\sqrt{1-(a+b)^2/4}$.

[Apply proof-by-contradiction method.
Ask. What to write at the beginning of the argument?
Answer. '*Let a, b be real numbers. Suppose $|a| \leq 1$ and $|b| \leq 1$. Suppose* $\sqrt{1-a^2} + \sqrt{1-b^2} > 2\sqrt{1-(a+b)^2/4}$.
Ask. What is our objective?
Answer. To obtain a contradiction out of what has been written down as assumption.]

Prove the statement $(\star)$ below:

$(\star)$ *Let a, b be real numbers. Suppose $|a| \leq 1$ and $|b| \leq 1$. Then* $\sqrt{1-a^2} + \sqrt{1-b^2} \leq 2\sqrt{1-(a+b)^2/4}$.

[Apply proof-by-contradiction method.]

Let a, b be real numbers. Suppose $|a| \leq 1$ and $|b| \leq 1$. Suppose $\sqrt{1-a^2} + \sqrt{1-b^2} > 2\sqrt{1-(a+b)^2/4}$.

[Ask. What is our objective?
Answer. To obtain a contradiction out of what has been written down as assumption.
Ask. How?
Further ask. It is inconvenient to work with three surd forms. Can we safely get rid of one (or more)?]

Prove the statement ($\star$) below:

($\star$) *Let* a, b *be real numbers. Suppose* $|a| \leq 1$ *and* $|b| \leq 1$*. Then* $\sqrt{1-a^2} + \sqrt{1-b^2} \leq 2\sqrt{1-(a+b)^2/4}$.

[Apply proof-by-contradiction method.]

Let a, b be real numbers. Suppose $|a| \leq 1$ and $|b| \leq 1$. Suppose $\sqrt{1-a^2} + \sqrt{1-b^2} > 2\sqrt{1-(a+b)^2/4}$.

[Ask. Is there something to fill in here? (We can wait.)]

$$\begin{aligned}(1-a^2)+(1-b^2)+2\sqrt{(1-a^2)(1-b^2)} &= (\sqrt{1-a^2}+\sqrt{1-b^2})^2 \\ &> \left[2\sqrt{1-\frac{(a+b)^2}{4}}\right]^2 = 4\left[1-\frac{(a+b)^2}{4}\right] = 4-(a+b)^2\end{aligned}$$

Therefore (after simplification), we would obtain $\sqrt{(1-a^2)(1-b^2)} > 1-ab$.

Ask. So what?
Answer. How about getting rid of the surd form?

Prove the statement $(\star)$ below:

$(\star)$ *Let a, b be real numbers. Suppose $|a| \leq 1$ and $|b| \leq 1$. Then $\sqrt{1-a^2} + \sqrt{1-b^2} \leq 2\sqrt{1-(a+b)^2/4}$.*

[Apply proof-by-contradiction method.]

Let a, b be real numbers. Suppose $|a| \leq 1$ and $|b| \leq 1$. Suppose $\sqrt{1-a^2} + \sqrt{1-b^2} > 2\sqrt{1-(a+b)^2/4}$.

[Ask. Is there something to fill in here? (We can wait.)]

$$\begin{aligned}(1-a^2) + (1-b^2) + 2\sqrt{(1-a^2)(1-b^2)} &= (\sqrt{1-a^2} + \sqrt{1-b^2})^2 \\ &> \left[2\sqrt{1 - \frac{(a+b)^2}{4}}\right]^2 = 4\left[1 - \frac{(a+b)^2}{4}\right] = 4 - (a+b)^2\end{aligned}$$

Therefore (after simplification), we would obtain $\sqrt{(1-a^2)(1-b^2)} > 1 - ab$.

[Ask. Is there something to fill in here? (We can wait.)]

Then $1 - a^2 - b^2 + a^2b^2 = (1-a^2)(1-b^2) > (1-ab)^2 = 1 - 2ab + a^2b^2$.

Therefore (after simplification), we would obtain $a^2 + b^2 - 2ab < 0$.

Prove the statement ($\star$) below:

($\star$) *Let* a, b *be real numbers. Suppose* $|a| \leq 1$ *and* $|b| \leq 1$*. Then* $\sqrt{1-a^2} + \sqrt{1-b^2} \leq 2\sqrt{1-(a+b)^2/4}$.

[Apply proof-by-contradiction method.]

Let a, b be real numbers. Suppose $|a| \leq 1$ and $|b| \leq 1$. Suppose $\sqrt{1-a^2} + \sqrt{1-b^2} > 2\sqrt{1-(a+b)^2/4}$.

[Ask. Is there something to fill in here? (We can wait.)]

$$\begin{aligned}(1-a^2)+(1-b^2)+2\sqrt{(1-a^2)(1-b^2)} &= (\sqrt{1-a^2}+\sqrt{1-b^2})^2 \\ &> \left[2\sqrt{1-\frac{(a+b)^2}{4}}\right]^2 = 4\left[1-\frac{(a+b)^2}{4}\right] = 4-(a+b)^2\end{aligned}$$

Therefore (after simplification), we would obtain $\sqrt{(1-a^2)(1-b^2)} > 1-ab$.

[Ask. Is there something to fill in here? (We can wait.)]

Then $1-a^2-b^2+a^2b^2 = (1-a^2)(1-b^2) > (1-ab)^2 = 1-2ab+a^2b^2$.
Therefore (after simplification), we would obtain $a^2+b^2-2ab < 0$.
But $a^2+b^2-2ab = (a-b)^2 \geq 0$ because a, b are real numbers. Contradiction arises.

Hence $\sqrt{1-a^2} + \sqrt{1-b^2} \leq 2\sqrt{1-\frac{(a+b)^2}{4}}$ in the first place.

Prove the statement ($\star$) below:

($\star$) *Let a, b be real numbers. Suppose $|a| \leq 1$ and $|b| \leq 1$. Then* $\sqrt{1-a^2} + \sqrt{1-b^2} \leq 2\sqrt{1-(a+b)^2/4}$.

[Apply proof-by-contradiction method.]

Let a, b be real numbers. Suppose $|a| \leq 1$ and $|b| \leq 1$. Suppose $\sqrt{1-a^2} + \sqrt{1-b^2} > 2\sqrt{1-(a+b)^2/4}$.

Note that $2\sqrt{1 - \dfrac{(a+b)^2}{4}} \geq 0$.

$$\begin{aligned}(1-a^2) + (1-b^2) + 2\sqrt{(1-a^2)(1-b^2)} &= (\sqrt{1-a^2} + \sqrt{1-b^2})^2 \\ &> \left[2\sqrt{1 - \frac{(a+b)^2}{4}}\right]^2 = 4\left[1 - \frac{(a+b)^2}{4}\right] = 4 - (a+b)^2\end{aligned}$$

Therefore (after simplification), we would obtain $\sqrt{(1-a^2)(1-b^2)} > 1 - ab$.
Since $|a| \leq 1$ and $|b| \leq 1$, we have $ab \leq |ab| \leq 1$. Then $1 - ab \geq 0$.
Then $1 - a^2 - b^2 + a^2b^2 = (1-a^2)(1-b^2) > (1-ab)^2 = 1 - 2ab + a^2b^2$.
Therefore (after simplification), we would obtain $a^2 + b^2 - 2ab < 0$.
But $a^2 + b^2 - 2ab = (a-b)^2 \geq 0$ because a, b are real numbers. Contradiction arises.

Hence $\sqrt{1-a^2} + \sqrt{1-b^2} \leq 2\sqrt{1 - \dfrac{(a+b)^2}{4}}$ in the first place.

6. Prove the statement ($\star$) below:

($\star$) *Let* $m, n \in \mathbb{N} \backslash \{0\}$. *Let* x *be a positive real number. Suppose* $m > n$. *Then* $x^m + \dfrac{1}{x^m} \geq x^n + \dfrac{1}{x^n}$. *Moreover, equality holds iff* $x = 1$.

Let $m, n \in \mathbb{N} \backslash \{0\}$. Let x be a positive real number. Suppose $m > n$.

[We want to deduce the inequality $x^m + \dfrac{1}{x^m} \geq x^n + \dfrac{1}{x^n}$.

Observations.

Recall: For any $u, v \in \mathbb{R}$, $u \geq v$ iff $u - v \geq 0$.

It is easier to see whether an expression 'appropriately factorized' as a product of 'simple factors' is non-negative or not.

Ask. Can we re-formulate the desired inequality as 'blah-blah-blah ≥ 0', and re-write 'blah-blah-blah' as a product of 'simple factors'?]

Prove the statement ($\star$) below:

($\star$) *Let $m, n \in \mathbb{N}\backslash\{0\}$. Let x be a positive real number. Suppose $m > n$. Then $x^m + \dfrac{1}{x^m} \geq x^n + \dfrac{1}{x^n}$. Moreover, equality holds iff $x = 1$.*

Let $m, n \in \mathbb{N}\backslash\{0\}$. Let x be a positive real number. Suppose $m > n$.

[We want to deduce the inequality $x^m + \dfrac{1}{x^m} \geq x^n + \dfrac{1}{x^n}$.
Ask. Can we re-formulate the desired inequality as 'blah-blah-blah ≥ 0', and re-write 'blah-blah-blah' as a product of 'simple factors'?]

We have $\left(x^m + \dfrac{1}{x^m}\right) - \left(x^n + \dfrac{1}{x^n}\right) = (x^m - x^n) - \dfrac{x^m - x^n}{x^{m+n}} = \dfrac{(x^m - x^n)(x^{m+n} - 1)}{x^{m+n}}$.

[So the desired inequality can be re-formulated as $\dfrac{(x^m - x^n)(x^{m+n} - 1)}{x^{m+n}} \geq 0$.
Ask. Can we tell whether this 'fraction' is non-negative or not?]

Prove the statement ($\star$) below:

($\star$) *Let $m, n \in \mathbb{N}\backslash\{0\}$. Let x be a positive real number. Suppose $m > n$. Then $x^m + \dfrac{1}{x^m} \geq x^n + \dfrac{1}{x^n}$. Moreover, equality holds iff $x = 1$.*

Let $m, n \in \mathbb{N}\backslash\{0\}$. Let x be a positive real number. Suppose $m > n$.

We have $\left(x^m + \dfrac{1}{x^m}\right) - \left(x^n + \dfrac{1}{x^n}\right) = (x^m - x^n) - \dfrac{x^m - x^n}{x^{m+n}} = \dfrac{(x^m - x^n)(x^{m+n} - 1)}{x^{m+n}}$.

[So the desired inequality can be re-formulated as $\dfrac{(x^m - x^n)(x^{m+n} - 1)}{x^{m+n}} \geq 0$. Does this inequality hold?]

Note that $x^{m+n} > 0$. We verify that $(x^m - x^n)(x^{m+n} - 1) \geq 0$:

[The form of the right-hand-side of such an inequality that we want to verify suggests we should 'split the argument' into two cases. Which two cases?
Focus on the expression $x^{m+n} - 1$. When is it negative? When is it non-negative?]

Prove the statement ($\star$) below:

($\star$) *Let $m, n \in \mathbb{N}\backslash\{0\}$. Let x be a positive real number. Suppose $m > n$. Then $x^m + \dfrac{1}{x^m} \geq x^n + \dfrac{1}{x^n}$. Moreover, equality holds iff $x = 1$.*

Let $m, n \in \mathbb{N}\backslash\{0\}$. Let x be a positive real number. Suppose $m > n$.

We have $\left(x^m + \dfrac{1}{x^m}\right) - \left(x^n + \dfrac{1}{x^n}\right) = (x^m - x^n) - \dfrac{x^m - x^n}{x^{m+n}} = \dfrac{(x^m - x^n)(x^{m+n} - 1)}{x^{m+n}}$.

Note that $x^{m+n} > 0$. We verify that $(x^m - x^n)(x^{m+n} - 1) \geq 0$.

- (Case 1). Suppose $0 < x < 1$. Then $0 < x^{m-n} < 1$ and $x^n > 0$.
  Therefore $x^m - x^n = (x^{m-n} - 1)x^n < 0$. Hence $x^m - x^n < 0$.
  Also $0 < x^{m+n} < 1$. Then $x^{m+n} - 1 < 0$. It follows that $(x^m - x^n)(x^{m+n} - 1) > 0$.
- (Case 2). Suppose $x \geq 1$. Then $x^{m-n} \geq 1$ and $x^n > 0$.
  Therefore $x^m - x^n = (x^{m-n} - 1) \cdot x^n > 0$. Hence $x^m - x^n \geq 0$.
  Also $x^{m+n} \geq 1$. Then $x^{m+n} - 1 \geq 0$. It follows that $(x^m - x^n)(x^{m+n} - 1) \geq 0$.

Therefore, in any case, $(x^m - x^n)(x^{m+n} - 1) \geq 0$.

It follows that $\left(x^m + \dfrac{1}{x^m}\right) - \left(x^n + \dfrac{1}{x^n}\right) = \dfrac{(x^m - x^n)(x^{m+n} - 1)}{x^{m+n}} \geq 0$. Hence $x^m + \dfrac{1}{x^m} \geq x^n + \dfrac{1}{x^n}$.

Prove the statement ($\star$) below:

($\star$) *Let $m, n \in \mathbb{N}\backslash\{0\}$. Let x be a positive real number. Suppose $m > n$. Then $x^m + \dfrac{1}{x^m} \geq x^n + \dfrac{1}{x^n}$. Moreover, equality holds iff $x = 1$.*

Let $m, n \in \mathbb{N}\backslash\{0\}$. Let x be a positive real number. Suppose $m > n$.

We have $\left(x^m + \dfrac{1}{x^m}\right) - \left(x^n + \dfrac{1}{x^n}\right) = (x^m - x^n) - \dfrac{x^m - x^n}{x^{m+n}} = \dfrac{(x^m - x^n)(x^{m+n} - 1)}{x^{m+n}}$.

[We now handle the matter of necessary and sufficient conditions for ‘equality’.]

- Suppose $x = 1$. Then $x^m + \dfrac{1}{x^m} = 2 = x^n + \dfrac{1}{x^n}$.

Prove the statement ($\star$) below:

($\star$) *Let* $m, n \in \mathbb{N}\backslash\{0\}$. *Let* x *be a positive real number. Suppose* $m > n$. *Then* $x^m + \dfrac{1}{x^m} \geq x^n + \dfrac{1}{x^n}$. *Moreover, equality holds iff* $x = 1$.

Let $m, n \in \mathbb{N}\backslash\{0\}$. Let x be a positive real number. Suppose $m > n$.

We have $\left(x^m + \dfrac{1}{x^m}\right) - \left(x^n + \dfrac{1}{x^n}\right) = (x^m - x^n) - \dfrac{x^m - x^n}{x^{m+n}} = \dfrac{(x^m - x^n)(x^{m+n} - 1)}{x^{m+n}}$.

[We now handle the matter of necessary and sufficient conditions for 'equality'.
Observation.
The inequality becomes an equality exactly when its re-formulation $\dfrac{(x^m - x^n)(x^{m+n} - 1)}{x^{m+n}} \geq 0$ becomes an equality.]

- Suppose $x = 1$. Then $x^m + \dfrac{1}{x^m} = 2 = x^n + \dfrac{1}{x^n}$.
- Suppose $x^m + \dfrac{1}{x^m} = x^n + \dfrac{1}{x^n}$. Then $0 = \dfrac{(x^m - x^n)(x^{m+n} - 1)}{x^{m+n}}$. Therefore $x^m = x^n$ or $x^{m+n} = 1$.
  - (Case 1). Suppose $x^m = x^n$. Then $x^{m-n} = 1$. Since $m > n$, we have $m - n > 0$. Since $x > 0$, we have $x = 1$.
  - (Case 2). Suppose $x^{m+n} = 1$. Note that $m + n > 0$. Since $x > 0$, we have $x = 1$.

  Hence, in any case, we have $x = 1$.