MATH1050 Exercise 10 Supplement

1. Define the relation $T = (\mathbb{R}, \mathbb{R}, G)$ in $\mathbb{R}$ by $G = \{(x, y) \in \mathbb{R}^2 : \text{There exists some } n \in \mathbb{Z} \text{ such that } y = 2^n x\}$.
   (a) Verify that T is reflexive.
   (b) Verify that T is transitive.
   (c) Verify that T is an equivalence relation in $\mathbb{R}$.

2. Let p be a positive real number. Define the relation $R = (\mathbb{C}, \mathbb{C}, E)$ in $\mathbb{C}$ by
$$E = \left\{(\zeta, \eta) \in \mathbb{C}^2 : \text{ There exists some } n \in \mathbb{Z} \text{ such that } \eta = \zeta \cdot (\cos(np) + i\sin(np)).\ \right\}$$
   (a) Verify that R is reflexive.
   (b) Verify that R is transitive.
   (c) Is R an equivalence relation in $\mathbb{C}$? Justify your answer.

3. Write $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Define the relation $R = (\mathbb{C}^*, \mathbb{C}^*, G)$ in $\mathbb{C}^*$ by
$$G = \left\{(\zeta, \eta) \in (\mathbb{C}^*)^2 : \text{ There exists some } n \in \mathbb{Z} \text{ such that } \zeta = \eta \cdot 2^n(\cos(n) + i\sin(n)).\ \right\}.$$
   (a) Verify that R is reflexive.
   (b) Verify that R is transitive.
   (c) Is R an equivalence relation in $\mathbb{C}^*$? Justify your answer.

4. Define the relation $T = (\mathbb{R}, \mathbb{R}, G)$ in $\mathbb{R}$ by
$G = \{(x, y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R} \text{ and (there exists some } m, n \in \mathbb{Q} \text{ such that } y = 3^m 5^n x)\}$.
   (a) Verify that T is reflexive.
   (b) Verify that T is transitive.
   (c) Verify that T is an equivalence relation in $\mathbb{R}$.

5.♣ Let A be a set, $G = \{(S, T) \mid S \in \mathfrak{P}(A) \text{ and } T \in \mathfrak{P}(A) \text{ and } S \subset T\}$ and $R = (\mathfrak{P}(A), \mathfrak{P}(A), G)$.
   (a) Verify that R is a partial ordering.
   (b) Suppose A has at least two distinct elements. Verify that R is not a total ordering.

6.♣ *Familiarity with the calculus of one variable is assumed in this question.*
   (a) Let A be the set of all real-valued continuous functions on $[0, 1]$. Define the relation $S = (A, A, G)$ in A by
$$G = \left\{(f, g) \in A^2 : \int_0^x uf(u)du \leq \int_0^x ug(u)du \text{ for any } x \in [0, 1]\right\}.$$
   Is S a partial ordering in A? Justify your answer.
   (b) Let B be the set of all real-valued piecewise-continuous functions on $[0, 1]$. Define the relation $T = (B, B, H)$ in B by
$$H = \left\{(f, g) \in B^2 : \int_0^x uf(u)du \leq \int_0^x ug(u)du \text{ for any } x \in [0, 1]\right\}.$$
   Is T a partial ordering in B? Justify your answer.

7.♡ Define the relation $S = (\mathbb{N}^2, \mathbb{N}^2, P)$ in $\mathbb{N}^2$ by $P = \left\{(u, v) \;\middle|\; \begin{array}{l}\text{There exist } m, n, p, q \in \mathbb{N} \text{ such that} \\ u = (m, n), v = (p, q) \text{ and } \dfrac{2n+1}{2^m} \leq \dfrac{2q+1}{2^p}\end{array}\right\}$.

   Here $\leq$ is the usual ordering in $\mathbb{R}$.
   (a) Verify that S is a partial ordering in $\mathbb{N}^2$.
   (b) Is S a total ordering in $\mathbb{N}^2$? Why?

8.♢ Define the relation $R = (\mathbb{C}, \mathbb{C}, P)$ by $P = \left\{(\zeta, \eta) \;\middle|\; \begin{array}{l}\zeta, \eta \in \mathbb{C} \text{ and} \\ (\mathsf{Re}(\zeta) < \mathsf{Re}(\eta) \text{ or } (\mathsf{Re}(\zeta) = \mathsf{Re}(\eta) \text{ and } \mathsf{Im}(\zeta) \leq \mathsf{Im}(\eta)))\end{array}\right\}$.
   (a) Let $\zeta, \eta \in \mathbb{C}$.
      i. Verify that $(\zeta, \eta) \in P$ iff $(\mathsf{Re}(\zeta) \leq \mathsf{Re}(\eta)$ and $(\mathsf{Re}(\zeta) < \mathsf{Re}(\eta)$ or $\mathsf{Im}(\zeta) \leq \mathsf{Im}(\eta)))$.
      ii. Verify that $(\zeta, \eta) \notin P$ iff $(\mathsf{Re}(\eta) < \mathsf{Re}(\zeta)$ or $(\mathsf{Re}(\eta) \leq \mathsf{Re}(\zeta)$ and $\mathsf{Im}(\eta) < \mathsf{Im}(\zeta)))$.
   (b) Verify that R is a total ordering in $\mathbb{C}$.

Remark. Such a total ordering in $\mathbb{C}$ is known as a **lexicographical ordering**. Think of each complex number as a word with two 'letters', the first 'letter' being its real part and the second 'letter' being its imaginary part respectively. Now how do you arrange such 'two-letter words' in a dictionary?

9. Denote by Σ the set of all infinite sequences in $\mathbb{R}$. (Recall that each infinite sequence in $\mathbb{R}$ is a function from $\mathbb{N}$ to $\mathbb{R}$.) Let $k \in \mathbb{N}$. Define the relation $R_k = (\Sigma, \Sigma, E)$ by

$$E = \left\{ (\alpha, \beta) \;\middle|\; \begin{array}{l} \alpha, \beta \in \Sigma \text{ and there exist some } N \in \mathbb{N},\, C \geq 0 \\ \text{such that } (|\alpha(x) - \beta(x)| \leq C/x^k \text{ for any } x \geq N). \end{array} \right\}$$

   (a)$^\diamondsuit$ Verify that R_k is reflexive and symmetric.

   (b)$^\clubsuit$ Verify that R_k is an equivalence relation in Σ.

10. (a) Let $A = \{0, 1, 2\}$, $G = \{(0,0), (1,1), (2,2), (0,1), (1,2)\}$, and $R = (A, A, G)$. (Here $0, 1, 2$ are pairwise distinct objects.)
      i. Verify that R is not symmetric.
      ii. Verify that R is not transitive.
      iii. Verify that R is reflexive.

    (b) Let $B = \{0, 1\}$, $H = \{(0,0), (0,1), (1,0)\}$, and $S = (B, B, H)$. (Here $0, 1$ are distinct objects.)
      i. Verify that S is not reflexive.
      ii. Verify that S is not transitive.
      iii. Verify that S is symmetric.

    (c) Let $C = \{0, 1, 2\}$, $J = \{(0,1), (1,2), (0,2)\}$, and $T = (C, C, J)$. (Here $0, 1, 2$ are pairwise distinct objects.)
      i. Verify that T is not reflexive.
      ii. Verify that T is not symmetric.
      iii. Verify that T is transitive.

Remark. Can you construct a relation in a non-empty set which is reflexive and symmetric but not transitive? Can you construct a relation in a non-empty set which is reflexive and transitive but not symmetric? Can you construct a relation in a non-empty set which is symmetric and transitive but not reflexive?

11.$^\diamondsuit$ Dis-prove each of the statements below by giving an appropriate counter-example.

   (a) *Let A be a non-empty set, and R be a relation in A. Suppose R is reflexive and symmetric. Then R is transitive.*

   (b) *Let A be a non-empty set, and R be a relation in A. Suppose R is reflexive and transitive. Then R is symmetric.*

   (c) *Let A be a non-empty set, and R be a relation in A. Suppose R is symmetric and transitive. Then R is reflexive.*

12. (a) Let A be a non-empty set, and R be a relation in A with graph G. Suppose R is symmetric and transitive. Prove that the statements below are logically equivalent:
      ($\sharp$) *For any $x \in A$, there exists some $y \in A$ such that $(x, y) \in G$.*
      ($\flat$) *R is reflexive.*

    (b) Let A be a non-empty set, and R be a relation in A with graph G. Suppose R is reflexive. Prove that the statements below are logically equivalent:
      ($\sharp$) *For any $x, y, z \in A$, if $(x, y) \in G$ and $(y, z) \in G$ then $(z, x) \in G$.*
      ($\flat$) *R is symmetric and transitive.*

    (c) Let A be a non-empty set, and R be a relation in A with graph G. Suppose R is reflexive. Prove that the statements below are logically equivalent:
      ($\sharp$) *For any $x, y, z \in A$, if $(x, y) \in G$ and $(x, z) \in G$ then $(y, z) \in G$.*
      ($\flat$) *R is symmetric and transitive.*

13.$^\diamondsuit$ Let A be a set, F be a subset of A^2, and $f = (A, A, F)$. Suppose f is a function from A to A. (Also think of f as a relation in A.) Prove the statements below:

   (a) *If f is reflexive as a relation in A then $f = \mathsf{id}_A$.*

   (b) *If f is transitive as a relation in A then $f \circ f = f$ as functions.*

   (c) *If f is transitive as a relation in A and f is injective as a function then $f = \mathsf{id}_A$.*

   (d) *If f is both symmetric and transitive as a relation in A then $f = \mathsf{id}_A$.*

14.$^\clubsuit$ We introduce the definition below:

   - *Let A, B be sets, $f : A \longrightarrow B$ be a function, and Q be a relation in B with graph H. Define the subset f^*H of A^2 by $f^*H = \{\ (x, w) \mid x \in A \text{ and } w \in A \text{ and } (f(x), f(w)) \in H\ \}$. The relation (A, A, f^*H) is called* **pull-back relation** *of Q by f. It is denoted by f^*Q in A.*

Let A, B be sets, $f : A \longrightarrow B$ be a function, and Q be a relation in B with graph H.
Prove the statements below:

(a) *Suppose Q is reflexive. Then f^*Q is reflexive.*
(b) *Suppose Q is symmetric. Then f^*Q is symmetric.*
(c) *Suppose Q is transitive. Then f^*Q is transitive.*
(d) *Suppose Q is an equivalence relation. Then f^*Q is an equivalence relation.*
(e) *Suppose f^*Q is an equivalence relation and f is surjective. Then Q is an equivalence relation.*
(f) *Suppose Q is reflexive and f^*Q is anti-symmetric. Then f is injective.*
(g) *Suppose Q is a partial ordering and f is injective. Then f^*Q is a partial ordering.*

15. Let A be a non-empty set, and R be a relation in A with graph E.
For any $x \in A$, we define $R[x] = \{y \in A : (x, y) \in E\}$. We define $\Omega = \{\ R[x] \ \mid x \in A\ \}$.
Suppose that R is an equivalence relation in A.

(a) Prove the statements below:

i. *For any $x \in A$, $x \in R[x]$.*
ii. $\emptyset \notin \Omega$.
iii.$^\diamondsuit$ *For any $x, y \in A$, if $(x, y) \in E$ then $R[y] \subset R[x]$.*
iv.$^\clubsuit$ *For any $x, y \in A$, the statements ($\sharp$), ($\natural$), ($\flat$) are logically equivalent:*

($\sharp$) $(x, y) \in E$. ($\natural$) $R[x] = R[y]$. ($\flat$) $R[x] \cap R[y] \neq \emptyset$.

Remark. $R[x]$ is called the **equivalence class** of x under the equivalence relation R.

(b)$^\clubsuit$ Apply part (a), or otherwise, to prove that Ω is a partition of A, in the sense that the statements (N), (U), (D) are true:

(N) $\emptyset \notin \Omega$.
(U) $\{z \in A : z \in S \text{ for some } S \in \Omega\} = A$
(D) For any $S, T \in \Omega$, exactly one of the statements '$S = T$', '$S \cap T = \emptyset$' is true.

Remark. We call Ω the **quotient** of A by the equivalence relation R, and usually write Ω as A/R. We refer to the elements of Ω as the equivalence classes under R.

(c)$^\heartsuit$ Let Φ be the subset of $A \times \Omega$ given by $\Phi = \{\ (x, S) \mid x \in A \text{ and } S \in \Omega \text{ and } x \in S\ \}$. Define the relation $\varphi = (A, \Omega, \Phi)$.

i. Prove that φ is a surjective function, and that $\varphi(x) = R[x]$ for any $x \in A$.
Remark. We call φ the **quotient mapping** of the equivalence relation R.
ii. Let B be a set and $f : A \longrightarrow B$ be a function. Suppose that for any $x, y \in A$, if $(x, y) \in E$ then $f(x) = f(y)$. Prove that there exists some unique function $g : \Omega \longrightarrow B$ such that $g \circ \varphi = f$.

16. Define the relation $R = (\mathbb{C}, \mathbb{C}, E)$ in $\mathbb{C}$ by $E = \{(\zeta, \eta) \in \mathbb{C}^2 : \mathsf{Re}(\zeta) = \mathsf{Re}(\eta)\}$.

(a) Verify that R is reflexive.
(b) Verify that R is symmetric.
(c) Verify that R is an equivalence relation in $\mathbb{C}$.
(d) For any $\zeta \in \mathbb{C}$, denote by $[\zeta]$ the equivalence class of ζ under R.
(Note that by definition, $[\zeta] = \{\eta \in \mathbb{C} : (\zeta, \eta) \in E\}$.)
What are the respective equivalence classes of 1, 0, i under R? Describe these sets in geometric terms in the Argand plane.

17. Write $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$.
Define the relation $R = (\mathbb{C}^*, \mathbb{C}^*, E)$ in $\mathbb{C}^*$ by $E = \left\{(\zeta, \eta) \in (\mathbb{C}^*)^2 : \dfrac{\mathsf{Re}(\zeta)}{|\zeta|^2} = \dfrac{\mathsf{Re}(\eta)}{|\eta|^2}\right\}$.

(a) Verify that R is an equivalence relation in $\mathbb{C}^*$.
(b)$^\clubsuit$ For any $\zeta \in \mathbb{C}^*$, denote by $[\zeta]$ the equivalence class of ζ under R.

i. Let $a \in \mathbb{R}^*$. Verify that $[ai] = \{ti \mid t \in \mathbb{R}^*\}$.
ii. Let $\zeta \in \mathbb{C}^*$. Suppose $\mathsf{Re}(\zeta) \neq 0$. Define $r_\zeta = \dfrac{|\zeta|^2}{2\mathsf{Re}(\zeta)}$. Verify the statements (†) and (‡):

(†) $(\zeta, 2r_\zeta) \in E$.
(‡) *Suppose $\eta \in \mathbb{C}^*$. Then $\eta \in [\zeta]$ iff* $(\mathsf{Re}(\eta) - r_\zeta)^2 + (\mathsf{Im}(\eta))^2 = (r_\zeta)^2$.

18. Define the relation $T = (\mathbb{C}, \mathbb{C}, G)$ in $\mathbb{C}$ by $G = \{(\zeta, \eta) \in \mathbb{C}^2 : \zeta^4 = \eta^4\}$.

(a) Verify that T is an equivalence relation in $\mathbb{C}$.

(b)$^\clubsuit$ For any $\zeta \in \mathbb{C}$, denote by $[\zeta]$ the equivalence class of ζ under T.
Prove the statements below:

i. *For any $\zeta, \eta \in \mathbb{C}$, if $\eta \in [\zeta]$ then ($\eta = \zeta$ or $\eta = i\zeta$ or $\eta = -\zeta$ or $\eta = -i\zeta$).*

ii. *For any $\zeta \in \mathbb{C}$, $[\zeta] = \{\zeta, i\zeta, -\zeta, -i\zeta\}$.*

(c)$^\heartsuit$ Denote by Ω the quotient of $\mathbb{C}$ by T, and define the function $\pi : \mathbb{C} \longrightarrow \Omega$ by $\pi(\zeta) = [\zeta]$ for any $\zeta \in \mathbb{C}$.
Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a function. Define

$$\varphi = \left\{ (U, \chi) \;\middle|\; \begin{array}{l} U \in \Omega \text{ and } \chi \in \mathbb{C} \text{ and} \\ \text{there exists } \zeta \in \mathbb{C} \text{ such that } U = [\zeta] \text{ and } \chi = f(\zeta^4). \end{array} \right\}.$$

Note that $\varphi \subset \Omega \times \mathbb{C}$.
Prove the statements below:

i. *φ is a function from Ω to $\mathbb{C}$.*

ii. *$(\varphi \circ \pi)(\zeta) = f(\zeta^4)$ for any $\zeta \in \mathbb{C}$.*

iii. *Let $\psi : \Omega \longrightarrow \mathbb{C}$ is a function. Suppose $(\psi \circ \pi)(\zeta) = f(\zeta^4)$ for any $\zeta \in \mathbb{C}$. Then $\psi = \varphi$.*

19. Let A, B be non-empty sets, and $f : A \longrightarrow B$ be a surjective function.
Define the relation $R_f = (A, A, E_f)$ in A by $E_f = \{(x, y) \mid x, y \in A \text{ and } f(x) = f(y)\}$.

(a) Verify that R_f is an equivalence relation.

(b)$^\clubsuit$ For any $x \in A$, denote the equivalence class of x under R_f by $[x]_f$.
Verify that $[x]_f = f^{-1}(\{f(x)\})$ for any $x \in A$.

(c)$^\clubsuit$ Define $\Omega = \{S \in \mathfrak{P}(A) \mid S = [x]_f \text{ for some } x \in A\}$.
Verify that Ω is a partition of A, in the sense that the statements (N), (U), (D) are true:

(N) $\emptyset \notin \Omega$.

(U) $\{z \in A : z \in S \text{ for some } S \in \Omega\} = A$.

(D) *For any $S, T \in \Omega$, exactly one of the statements '$S = T$', '$S \cap T = \emptyset$' is true.*

(d)$^\clubsuit$ Define $G_f = \{(x, S) \mid x \in A \text{ and } S \in \Omega \text{ and } x \in S\}$ and $\pi_f = (A, \Omega, G_f)$.
Verify that π_f is a surjective function.

(e)$^\spadesuit$ Let $\varphi : A \longrightarrow C$ be a function. Suppose that for any $x, y \in A$, if $f(x) = f(y)$ then $\varphi(x) = \varphi(y)$. Prove that there exists some unique function $\psi : \Omega \longrightarrow C$ such that $\psi \circ \pi = \varphi$.

20.$^\heartsuit$ Recall that whenever $n \in \mathbb{N}\backslash\{0, 1\}$, the relation $R_n = (\mathbb{Z}, \mathbb{Z}, E_n)$ given by $E_n = \{(x, y) \mid x, y \in \mathbb{Z} \text{ and } x \equiv y (\text{mod } n)\}$ is an equivalence relation in $\mathbb{Z}$. The quotient of $\mathbb{Z}$ by R_n is the set $\mathbb{Z}_n$.

For each $x \in \mathbb{Z}$, we denote by $[x]_n$ the equivalence class of x under the equivalence relation R_n in $\mathbb{Z}$. It is the element of $\mathbb{Z}_n$ given explicitly by $[x]_n = \{x \in \mathbb{Z} : (x, y) \in E_n\} = \{x \in \mathbb{Z} : x \equiv y (\text{mod } n)\}$.

Below are several 'declarations' through each of which some function is supposed to be defined. Determine whether it makes sense or not. Justify your answer.

(a) '*Define the function $f : \mathbb{Z}_{10} \longrightarrow \mathbb{Z}$ by $f([k]_{10}) = 10k$ for any $k \in \mathbb{Z}$.*'

(b) '*Define the function $f : \mathbb{Z}_{10} \longrightarrow \mathbb{Z}_{100}$ by $f([k]_{10}) = [k]_{100}$ for any $k \in \mathbb{Z}$.*'

(c) '*Define the function $f : \mathbb{Z}_{100} \longrightarrow \mathbb{Z}_{10}$ by $f([k]_{100}) = [k]_{10}$ for any $k \in \mathbb{Z}$.*'

(d) '*Define the function $f : \mathbb{Z}_{10} \longrightarrow \mathbb{Z}_{100}$ by $f([k]_{10}) = [10k]_{100}$ for any $k \in \mathbb{Z}$.*'

(e) '*Define the function $f : \mathbb{Z}_{10} \longrightarrow \mathbb{Z}_{10}$ by $f([k]_{10}) = [3k]_{10}$ for any $k \in \mathbb{Z}$.*'

(f) '*Define the function $f : \mathbb{Z}_{10} \longrightarrow \mathbb{Z}_{10}$ by $f([3k]_{10}) = [k]_{10}$ for any $k \in \mathbb{Z}$.*'

(g) '*Define the function $f : \mathbb{Z}_{10} \longrightarrow \mathbb{Z}_{10}$ by $f([4k]_{10}) = [3k]_{10}$ for any $k \in \mathbb{Z}$.*'

21.$^\spadesuit$ Let $\mathbb{G} = \{\zeta \in \mathbb{C} : \mathsf{Re}(\zeta) \in \mathbb{Z} \text{ and } \mathsf{Im}(\zeta) \in \mathbb{Z}\}$. ($\mathbb{G}$ is the set of all Gaussian integers.)
Define the subset E of $\mathbb{C}^2$ by $E = \{(\zeta, \eta) \mid \zeta, \eta \in \mathbb{C} \text{ and } \zeta - \eta \in \mathbb{G}\}$.
Define $R = (\mathbb{C}, \mathbb{C}, E)$.
For each $\zeta \in \mathbb{C}$, define $[\zeta] = \{\eta \in \mathbb{C} : (\zeta, \eta) \in E\}$.
Let $T = \{[\zeta] \mid \zeta \in \mathbb{C}\}$.
Throughout this question, you may take the validity of the statements **(S1)**, **(S2)**, **(S3)** *for granted:*

(S1) *R is an equivalence relation in $\mathbb{C}$.*

(S2) *For any $\zeta \in \mathbb{C}$, $\zeta \in [\zeta]$.*

(S3) *For any $\zeta, \eta \in \mathbb{C}$, the statements ($\sharp$), ($\natural$), ($\flat$) are equivalent:*
($\sharp$) $(\zeta, \eta) \in E$. ($\natural$) $[\zeta] = [\eta]$. ($\flat$) $[\zeta] \cap [\eta] \neq \emptyset$.

(a) Define the subset Σ of $T^2 \times T$ by

$$\Sigma = \left\{ ((p, q), r) \;\middle|\; \begin{array}{l} p, q, r \in T \text{ and (there exist some } \zeta, \eta \in \mathbb{C} \\ \text{such that } p = [\zeta], q = [\eta] \text{ and } r = [\zeta + \eta]). \end{array} \right\}.$$

Define $\alpha = (T^2, T, \Sigma)$. Note that α is a relation from T^2 to T.
Verify that α is a function from T^2 to T.

(b) Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a surjective function. Consider the statements $(\star), (\star\star)$ below:
- $(\star)$ *There exists some surjective function $h : T \longrightarrow T$ such that for any $\zeta \in \mathbb{C}$, $h([\zeta]) = [f(\zeta)]$.*
- $(\star\star)$ *For any $\zeta, \eta \in \mathbb{C}$, if $\zeta - \eta \in \mathbb{G}$ then $f(\zeta) - f(\eta) \in \mathbb{G}$.*

i. Suppose $(\star)$ holds. Prove that $(\star\star)$ holds.
ii. Suppose $(\star\star)$ holds. Prove that $(\star)$ holds.

22.♠ Let $\lambda \in \mathbb{C}\backslash\{0\}$.

Define the subset E of $\mathbb{C}^2$ by $E = \{(\zeta, \eta) \in \mathbb{C}^2 : \mathsf{Re}(\overline{\lambda}\zeta) = \mathsf{Re}(\overline{\lambda}\eta)\}$.
Define $R = (\mathbb{C}, \mathbb{C}, E)$.
For each $\zeta \in \mathbb{C}$, define $[\zeta] = \{\eta \in \mathbb{C} : (\zeta, \eta) \in E\}$.
Let $L = \{[\zeta] \mid \zeta \in \mathbb{C}\}$.
Throughout this question, you may take the validity of the statements **(S1)**, **(S2)**, **(S3)** *for granted:*

(S1) *R is an equivalence relation in $\mathbb{C}$.*
(S2) *For any $\zeta \in \mathbb{C}$, $\zeta \in [\zeta]$.*
(S3) *For any $\zeta, \eta \in \mathbb{C}$, the statements ($\sharp$), ($\natural$), ($\flat$) are equivalent:*
($\sharp$) $(\zeta, \eta) \in E$. ($\natural$) $[\zeta] = [\eta]$. ($\flat$) $[\zeta] \cap [\eta] \neq \emptyset$.

(a) Define the subset Σ of $L^2 \times L$ by

$$\Sigma = \left\{ ((p, q), r) \;\middle|\; \begin{array}{l} p, q, r \in L \text{ and (there exist some } \zeta, \eta \in \mathbb{C} \\ \text{such that } p = [\zeta], q = [\eta] \text{ and } r = [\zeta + \eta]). \end{array} \right\}.$$

Define $\alpha = (L^2, L, \Sigma)$. Note that α is a relation from L^2 to L.
Verify that α is a function from L^2 to L.

(b) Now also suppose $\mathsf{Re}(\lambda) \neq 0$. Define the function $f : \mathbb{C} \longrightarrow \mathbb{R}$ by

$$f(\zeta) = \frac{\mathsf{Re}(\overline{\lambda}\zeta)}{\mathsf{Re}(\lambda)} \quad \text{for any } \zeta \in \mathbb{C}.$$

Prove the statement $(\star)$:
- $(\star)$ *There exists some bijective function $h : L \longrightarrow \mathbb{R}$ such that (for any $\zeta \in \mathbb{C}$, $h([\zeta]) = f(\zeta)$) and (for any $\sigma, \tau \in \mathbb{C}$, $h(\alpha([\sigma], [\tau])) = f(\sigma) + f(\tau)$).*

23.♠ Write $\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$.

Define the subset F of $(\mathbb{Z} \times \mathbb{Z}^*)^2$ by

$$F = \{((x, y), (x', y')) \mid x, x' \in \mathbb{Z} \text{ and } y, y' \in \mathbb{Z}^* \text{ and } xy' = x'y \}.$$

Define $Q = (\mathbb{Z} \times \mathbb{Z}^*, \mathbb{Z} \times \mathbb{Z}^*, F)$
For any $x \in \mathbb{Z}, y \in \mathbb{Z}^*$, define $[x, y] = \{(s, t) \mid s \in \mathbb{Z} \text{ and } t \in \mathbb{Z}^* \text{ and } ((x, y), (s, t)) \in F\}$.
Let $\Phi = \{[x, y] \mid x \in \mathbb{Z} \text{ and } y \in \mathbb{Z}^*\}$.
Throughout this question, you may take the validity of the statements **(S1)**, **(S2)**, **(S3)** *for granted:*

(S1) *Q is an equivalence relation in $\mathbb{Z} \times \mathbb{Z}^*$.*
(S2) *For any $x \in \mathbb{Z}$, for any $y \in \mathbb{Z}^*$, $(x, y) \in [(x, y)]$.*
(S3) *For any $x, x' \in \mathbb{Z}$, for any $y, y' \in \mathbb{Z}^*$, the statements ($\sharp$), ($\natural$), ($\flat$) are equivalent:*
($\sharp$) $((x, y), (x', y')) \in F$. ($\natural$) $[x, y] = [x', y']$. ($\flat$) $[x, y] \cap [x', y'] \neq \emptyset$.

(a) Define the subset G of $\Phi^2 \times \Phi$ by

$$G = \left\{ ((u,v),w) \;\middle|\; \begin{array}{l} \text{There exist some } x, x' \in \mathbb{Z}, y, y' \in \mathbb{Z}^* \\ \text{such that } u = [x,y] \text{ and } v = [x',y'] \text{ and } w = [xy' + yx', yy']. \end{array} \right\}.$$

Define $\alpha = (\Phi^2, \Phi, G)$. Note that α is a relation from G^2 to G.
Verify that α is a function.

(b) For any $u, v \in \Phi$, we write $\alpha(u,v)$ as $u \oplus v$.
Verify the statements below:

i. *For any* $u, v \in \Phi$, $u \oplus v = v \oplus u$.
ii. *For any* $u, v, w \in \Phi$, $(u \oplus v) \oplus w = u \oplus (v \oplus w)$.
iii. *There exists some unique* $e \in \Phi$ *such that for any* $u \in \Phi$, $u \oplus e = u$ *and* $e \oplus u = u$.
iv. *For any* $u \in \Phi$, *there exists some unique* $v \in \Phi$ *such that* $u \oplus v = e$ *and* $v \oplus u = e$. (Here e is the unique element of Φ which satisfies $u \oplus e = u = e \oplus u$ for any $u \in \Phi$.)

24. (a) Verify that $2^x(2y+1) \in \mathbb{N}\backslash\{0\}$ for any $x, y \in \mathbb{N}$.
(b) Define the function $f : \mathbb{N}^2 \longrightarrow \mathbb{N}\backslash\{0\}$ by $f(x,y) = 2^x(2y+1)$ for any $x, y \in \mathbb{N}$.
Verify that f is bijective.
(c) Verify that $\mathbb{N}^2 \sim \mathbb{N}$.

25. Let $S = \{x \in \mathbb{N} : x = m^2 \text{ for some } m \in \mathbb{N}\}$, $C = \{y \in \mathbb{N} : y = n^3 \text{ for some } n \in \mathbb{N}\}$.

Define $F = \left\{ (x,y) \;\middle|\; \begin{array}{l} x \in S \text{ and } y \in C \text{ and} \\ \text{there exists some } k \in \mathbb{N} \\ \text{such that } (x = k^2 \text{ and } y = k^3). \end{array} \right\}$, and $f = (S, C, F)$. Note that $F \subset S \times C$.

(a) Is f a function from S to C? Justify your answer.
(b) Is it true that $S \sim C$? Justify your answer.

26. Let p, q be distinct positive odd integers, and

$$A = \{x \in \mathbb{Q} : x = s^p \text{ for some } s \in \mathbb{Q}\}, \quad B = \{y \in \mathbb{Q} : y = t^q \text{ for some } t \in \mathbb{Q}\}$$

Define $F = \left\{ (x,y) \;\middle|\; \begin{array}{l} x \in A \text{ and } y \in B \text{ and} \\ \text{there exists some } r \in \mathbb{Q} \\ \text{such that } (x = r^p \text{ and } y = r^q). \end{array} \right\}$ and $f = (A, B, F)$. Note that $F \subset A \times B$.

(a)$^\diamondsuit$ Is f a function from A to B? Justify your answer.
(b) Is it true that A is of cardinality equal to B? Justify your answer.

27. (a) Let $A_1 = [1,2]$, $B_1 = (3,4)$. Apply the Schröder-Bernstein Theorem to prove that $A_1 \sim B_1$.
(b) Let $A_2 = [0, +\infty)$, $B_2 = (-1,1) \cup [2,3]$. Apply the Schröder-Bernstein Theorem to prove that $A_2 \sim B_2$.
(c)$^\diamondsuit$ Let $A_3 = (-\infty, -1) \cup \mathbb{N}$, $B_3 = [0.1, 0.9] \cup (1.1, 1.9)$. Apply the Schröder-Bernstein Theorem to prove that $A_3 \sim B_3$.
(d)$^\diamondsuit$ Let $A_4 = [1,9] \cup (\mathbb{Q} \cap [10,99])$, $B_4 = (0.01, 0.09) \cup (0.1, 0.9) \cup \mathbb{N}$. Apply the Schröder-Bernstein Theorem to prove that $A_4 \sim B_4$.
(e)$^\diamondsuit$ Let $A_5 = [1,2] \cup \{100\}$ and $B_5 = (1,10) \cup ((100,+\infty)\backslash\mathbb{Q})$. Apply the Schröder-Bernstein Theorem to prove that $A_5 \sim B_5$.
(f)$^\clubsuit$ Let $D = \{\zeta \in \mathbb{C} \mid |\zeta| \leq 1\}$, $S = \{\zeta \in \mathbb{C} : |\mathsf{Re}(\zeta)| \leq 1 \text{ and } |\mathsf{Im}(\zeta)| \leq 1\}$. Apply the Schröder-Bernstein Theorem to prove that $D \sim S$.

28.$^\clubsuit$ *In this question, you may take for granted the results* $[0,1] \sim \mathbb{R}$, $[0,1] \sim [0,1]^2$, $\mathbb{R} \sim \mathbb{R}^2$.

(a) Let Π be the set of all planes in $\mathbb{R}^3$. Apply the Schröder-Bernstein Theorem to prove that $\Pi \sim \mathbb{R}$.
Remark. Let Λ be the set of all lines in $\mathbb{R}^3$. How to prove $\Lambda \sim \mathbb{R}$?
(b) Let $\mathbb{S}^2 = \{(x,y,z) \mid x,y,z \in \mathbb{R} \text{ and } x^2+y^2+z^2 = 1\}$, $\mathbb{B}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R} \text{ and } x^2+y^2+z^2 \leq 1\}$. Apply the Schröder-Bernstein Theorem to prove that $\mathbb{S}^2 \sim \mathbb{B}^3$.
(c) Let $\mathbb{S}^1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$, $\mathbb{S}^2 = \{(x,y,z) \mid x,y,z \in \mathbb{R} \text{ and } x^2+y^2+z^2 = 1\}$. Apply the Schröder-Bernstein Theorem to prove that $\mathbb{S}^1 \sim \mathbb{S}^2$.

29. Let A, B be non-empty sets. Suppose each of A, B is not a singleton. Pick $a, a' \in A$, with $a \neq a'$, and pick $b, b' \in B$, with $b \neq b'$. Regard $0, 1$ as distinct objects.

(a) Construct an injective function from $A \cup B$ to $(A \times \{0\}) \cup (B \times \{1\})$.
(b) Construct a bijective function from $A \times \{0\}$ to $A \times \{b\}$.

(c) Construct a bijective function from $B \times \{1\}$ to $(\{a\} \times (B\backslash\{b\})) \cup \{(a', b')\}$.

(d) Construct a bijective function from $(A \times \{0\}) \cup (B \times \{1\})$ to $(A \times \{b\}) \cup (\{a\} \times (B\backslash\{b\})) \cup \{(a', b')\}$.

(e) Conclude that $A \cup B \lesssim A \times B$.

30.$^\heartsuit$ *In this question, we are going to give a proof for the Schröder-Bernstein Theorem.*

(a) Let A, B be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow A$ be injective functions.
For any subset V of B, define $V^* = B\backslash f(A\backslash g(V))$. (Note that V^* is a subset of B.)
Define $\mathfrak{C} = \{V \in \mathfrak{P}(B) : V^* \subset V\}$, $K = \{y \in B : y \in V$ for any $V \in \mathfrak{C}\}$.
Prove the statements below:

i. *For any subsets V, W of B, if $V \subset W$ then $V^* \subset W^*$.*

ii. $K \in \mathfrak{C}$.
Remark. This is a hint: *By the definition of K, we have $K \subset W$ for any $W \in \mathfrak{C}$.*

iii. $K^* = K$.

iv. $f(A\backslash g(K)) = B\backslash K$.

(b) Apply the above results to prove the Schröder-Bernstein Theorem.
Remark. How to start the argument? Focus on what part (a.iv) suggests for a pair of injective functions whose respective domains are the respective ranges of the others. At some stage of the subsequent argument, you may need the Glueing Lemma.

31. (a) Define the function $\Phi : \mathsf{Map}(\mathbb{N}, \{0, 1\}) \longrightarrow \mathsf{Map}(\mathbb{N}, \{0, 1, 2\})$ by $(\Phi(\alpha))(x) = \alpha(x)$ for any $x \in \mathbb{N}$.
Verify that Φ is an injective function.

(b)$^\clubsuit$ Apply the Schröder-Bernstein Theorem, or otherwise, to prove that $\mathsf{Map}(\mathbb{N}, \{0, 1\}) \sim \mathsf{Map}(\mathbb{N}, \{0, 1, 2\})$.

32. (a) Let A, B, C, D be non-empty sets. Prove the statements below:

i.$^\clubsuit$ *Suppose $A \sim C$ and $B \sim D$. Then $\mathsf{Map}(A, B) \sim \mathsf{Map}(C, D)$.*

ii.$^\diamondsuit$ *Suppose $A \subset C$. Then $\mathsf{Map}(A, B) \lesssim \mathsf{Map}(C, B)$.*

iii.$^\diamondsuit$ *Suppose $B \subset D$. Then $\mathsf{Map}(A, B) \lesssim \mathsf{Map}(A, D)$.*

iv.$^\diamondsuit$ *Suppose $B \lesssim D$. Then $\mathsf{Map}(A, B) \lesssim \mathsf{Map}(A, D)$.*

v.$^\diamondsuit$ *Suppose $A \subset C$ and $B \subset D$. Then $\mathsf{Map}(A, B) \lesssim \mathsf{Map}(C, D)$.*

vi.$^\heartsuit$ $\mathsf{Map}(A \times B, C) \sim \mathsf{Map}(A, \mathsf{Map}(B, C))$.

(b)$^\heartsuit$ Prove each of the statements below. Where necessary, apply the Schröder-Bernstein Theorem. You may take for granted that $\mathbb{N}^2 \sim \mathbb{N}$, $\mathbb{R}^2 \sim \mathbb{R}$ and $\mathbb{R} \sim \mathsf{Map}(\mathbb{N}, [\![0, 9]\!])$.

i. $\mathsf{Map}(\mathbb{N}, \{0, 1\}) \lesssim \mathsf{Map}(\mathbb{N}, \mathbb{N})$.

ii. $\mathsf{Map}(\mathbb{N}, \mathbb{N}) \lesssim \mathsf{Map}(\mathbb{N}, \mathsf{Map}(\mathbb{N}, \{0, 1\}))$.

iii. $\mathsf{Map}(\mathbb{N}, \mathbb{N}) \sim \mathsf{Map}(\mathbb{N}, \{0, 1\})$.

iv. $\mathbb{R} \sim \mathsf{Map}(\mathbb{N}, \mathbb{N})$.

v. $\mathsf{Map}(\mathbb{R}, \{0, 1\}) \sim \mathsf{Map}(\mathbb{R}, \mathbb{N})$.

vi. $\mathsf{Map}(\mathbb{R}, \mathbb{N}) \sim \mathsf{Map}(\mathbb{R}, \mathbb{R})$.

33.$^\heartsuit$ We introduce/recall the definitions below:

- *Let $z \in \mathbb{C}$.*
  - *z is said to be a* **Gaussian rational number** *if both of $\mathsf{Re}(z), \mathsf{Im}(z)$ are rational numbers.*
  - *z is said to be a* **Gaussian irrational number** *if z is not a Gaussian rational number.*

The set of all Gaussian rational numbers is denoted by $\mathbb{Q}[i]$.

For any $p, q \in \mathbb{C}$, we define $\sigma[p, q]$ to be the set $\{\tau p + (1 - \tau)q \mid \tau \in [0, 1]\}$. ($\sigma[p, q]$ is the line segment on the Argand plane joining the point p and the point q.)

Let $z_1, z_2 \in \mathbb{C}\backslash\mathbb{Q}[i]$. Suppose $z_1 \neq z_2$. Prove that there exist some $w \in \mathbb{C}\backslash\mathbb{Q}[i]$ such that the $\sigma[z_1, w] \cup \sigma[z_2, w] \subset \mathbb{C}\backslash\mathbb{Q}[i]$.

Remark. Hence any two Gaussian irrational numbers can be joint by a path made up of two line segments which lie entirely in the set of Gaussian irrational numbers. The proof-by-contradiction method is more suitable for the argument for this result. At some stage of the argument you may need the result $\mathbb{N} < \mathbb{R}$ (or something equivalent) and the Schröder-Bernstein Theorem.

34.$^\heartsuit$ *Familiarity with the calculus of one variable is assumed in this question.*

Let J be an open interval in $\mathbb{R}$. Denote by $C(J)$ the set of all real-valued continuous functions on J. Denote by $C^1(J)$ the set of all real-valued differentiable functions on J whose first derivatives are continuous functions on J.

Apply the Schröder-Bernstein Theorem, or otherwise, to prove that $C(J) \sim C^1(J)$.

35.$^\heartsuit$ Consider the sets $\mathbb{N}$ and $\mathfrak{P}(\mathbb{N})$. We introduce these notations:

- *We write $\mathfrak{F}(\mathbb{N}) = \{S \in \mathfrak{P}(\mathbb{N}) : S$ is finite.$\}$.* ($\mathfrak{F}(\mathbb{N})$ is the set of all finite subsets of $\mathbb{N}$.)

- *For any* $n \in \mathbb{N}$, *we write* $\mathfrak{F}_n(\mathbb{N}) = \{S \in \mathfrak{P}(\mathbb{N}) : S \text{ is finite and } |S| = n.\}$. ($\mathfrak{F}_n(\mathbb{N})$ is the set of all subsets of cardinality n of $\mathbb{N}$. It is by definition a subset of $\mathfrak{F}(\mathbb{N})$.)
- *We write* $\mathfrak{C}_\infty(\mathbb{N}) = \{S \in \mathfrak{P}(\mathbb{N}) : S \text{ is countably infinite.}\}$. ($\mathfrak{C}_\infty(\mathbb{N})$ is the set of all countably infinite subsets of $\mathbb{N}$.)

Note that the statements below hold:

(A) $\mathfrak{F}(\mathbb{N}) \cup \mathfrak{C}_\infty(\mathbb{N}) = \mathfrak{P}(\mathbb{N})$.

(B) $\mathfrak{F}(\mathbb{N}) \cap \mathfrak{C}_\infty(\mathbb{N}) = \emptyset$.

(C) $\mathfrak{F}(\mathbb{N}) = \{S \in \mathfrak{F}(\mathbb{N}) : S \in \mathfrak{F}_n(\mathbb{N}) \text{ for some } n \in \mathbb{N}\}$.

(D) $\mathfrak{F}_m(\mathbb{N}) \cap \mathfrak{F}_n(\mathbb{N}) = \emptyset$ *whenever* $m \neq n$.

These combine together to give the formal formulation of the 'fact' that $\mathfrak{P}(\mathbb{N})$ is 'partitioned' into these 'infinitely many' 'chambers': the set of all (countably) infinite subsets of $\mathbb{N}$, the set of all (finite) subsets of $\mathbb{N}$ with one element, the set of all (finite) subsets of $\mathbb{N}$ with two elements, the set of all (finite) subsets of $\mathbb{N}$ with three elements, ...

(a) What is $\mathfrak{F}_0(\mathbb{N})$?

(b) Write down a bijective function from $\mathbb{N}$ to $\mathfrak{F}_1(\mathbb{N})$.

(c) Write down a surjective function from $\mathbb{N}^2$ to $\mathfrak{F}_2(\mathbb{N}) \cup \mathfrak{F}_1(\mathbb{N})$.

(d) Is there an injective function from $\mathfrak{F}_2(\mathbb{N})$ to $\mathbb{N}^2$? Justify your answer.

(e) Is there an injective function from $\mathfrak{F}_3(\mathbb{N})$ to $\mathbb{N}^3$? Justify your answer.

(f) Is it true that $\mathfrak{F}_n(\mathbb{N})$ is countable for any $n \in \mathbb{N}$? Justify your answer.

(g) Is it true that $\mathfrak{F}(\mathbb{N})$ is countable? Justify your answer.

(h) Is $\mathfrak{C}_\infty(\mathbb{N})$ countable? Justify your answer.

36.♠ Let A be a non-empty finite set. We introduce these notations:

- *We write* $\mathfrak{S}(A) = \bigcup_{n=0}^{\infty} \mathsf{Map}([\![1, n]\!], A)$. ($\mathfrak{S}(A)$ is the set of all finite sequences in A. Read $\bigcup_{n=0}^{\infty} \mathsf{Map}([\![1, n]\!], A)$ as $\{\varphi \mid \varphi \in \mathsf{Map}([\![1, n]\!], A) \text{ for some } n \in \mathbb{N}\}$.)
- *For any* $n \in \mathbb{N}$, *we write* $\mathfrak{S}_n(A) = \mathsf{Map}([\![1, n]\!], A)$. ($\mathfrak{S}_n(A)$ is the set of all finite sequences of length n in A.)

(a) Let $n \in \mathbb{N}$. Is $\mathfrak{S}_n(A)$ finite? If it is finite, what is its cardinality?

(b) Is $\mathfrak{S}(A)$ countably infinite? Why?

(c) Is there any surjective function from $\mathfrak{S}(A)$ to $\mathsf{Map}(\mathfrak{S}(A), \mathfrak{S}(A))$? Why?