MATH1050 Exercise 10

1. *You are not required to justify your answer in this question. You are only required to give one correct answer for each ordered pair, although there may be different correct answers.*

   Let $A = [0, +\infty)$ and G, H be the subsets of $\mathbb{R}^2$ defined respectively by $G = \{(x, x) \mid x > 0\}$, $H = \{(x, y) \mid x \geq 0 \text{ and } y > 0 \text{ and } x^2 + y^2 = 1\}$.

   Name some appropriate ordered pairs $(s, t), (u, v) \in A^2$, if such exist, for which the ordered triple $(A, A, (G \cup H \cup \{(s, t), (u, v)\}))$ is a reflexive and symmetric relation in A.

2. Define the relation $S = (\mathbb{R}, \mathbb{R}, E)$ by $E = \{(x, y) \in \mathbb{R}^2 : x - y = a \text{ for some } a \in \mathbb{Q}\}$.

   Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement '*S is an equivalence relation in* $\mathbb{R}$'. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

- [We want to verify the reflexivity of S.]

We verify the statement ' ______(I)______ ':

______(II)______ .

Note that ______(III)______ , and $0 \in \mathbb{Q}$. Then ______(IV)______ , by the definition of S.

It follows that S is reflexive.

- [We want to verify the symmetry of S.]

We verify the statement ' ______(V)______ ':

Pick any $x, y \in \mathbb{R}$. ______(VI)______ .

Then ______(VII)______ such that $x - y = a$.

Note that ______(VIII)______ .

Since ______(IX)______ , we have $-a \in \mathbb{Q}$.

Now we have $y - x = -a$ and ______(X)______ . Then ______(XI)______ , by the definition of S.

It follows that S is symmetric.

- [We want to verify the transitivity of S.]

We verify the statement ' ______(XII)______ ':

Pick any ______(XIII)______ . Suppose ______(XIV)______ .

Since $(x, y) \in E$, there exists some $a \in \mathbb{Q}$ such that ______(XV)______ .

______(XVI)______ .

Note that $x - z = (x - y) + (y - z) = a + b$.

______(XVII)______ .

Now we have $x - z = a + b$, and $a + b \in \mathbb{Q}$. Then ______(XVIII)______ , by the definition of S.

It follows that S is transitive.

Since S is reflexive, symmetric and transitive, S is an equivalence relation.

3. Define the relation $S = (\mathbb{R}, \mathbb{R}, G)$ by $G = \{(x, y) \in \mathbb{R}^2 : x - y > xy\}$.

   Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement '*S is not reflexive, and S is not symmetric, and S is not transitive*'. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

- [We verify that S is not reflexive.]

We verify the statement ‘ ______(I)______ ’:

______(II)______ . Note that $x_0 \in \mathbb{R}$.

Also note that ____(III)____ $= 0 \leq 0 =$ ____(IV)____ . Therefore ‘$x_0 - x_0 > x_0 \cdot x_0$’ is false.

Hence ____(V)____ .

It follows that S is not reflexive.

- [We verify that S is not symmetric.]

We verify the statement ‘ ________(VI)________ ’:

Take ______(VII)______ . Note that $x_0, y_0 \in \mathbb{R}$.

Note that $x_0 - y_0 =$ ____(VIII)____ $= x_0 y_0$. Then ____(IX)____ .

Also note that $y_0 - x_0 = -1 \leq 0 = y_0 x_0$. Then ‘ ______(X)______ ’ is false.

Therefore $(y_0, x_0) \notin G$.

Hence for the same $x_0, y_0 \in \mathbb{R}$, ________(XI)________ simultaneously.

It follows that S is not symmetric.

- [We verify that S is not transitive.]

We verify the statement ‘ ________(XII)________ ’:

Take $x_0 = -2$, $y_0 = 3$, ____(XIII)____ . Note that $x_0, y_0, z_0 \in \mathbb{R}$.

Note that ______(XIV)______ Then $(x_0, y_0) \in G$.

Also note that $y_0 - z_0 = 5 > -6 = y_0 z_0$. Then ____(XV)____ .

Finally, note that ______(XVI)______ . Then ‘$x_0 - z_0 > x_0 z_0$’ is false.

Therefore ____(XVII)____ .

Hence for the same $x_0, y_0, z_0 \in \mathbb{R}$, ________(XVIII)________ simultaneously.

It follows that S is not transitive.

4. *You are not required to justify your answer in this question.*
   Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
   Which of the sets below are partitions of A? Which not?

$$\Omega = \{\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}\}, \qquad \Xi = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}\},$$
$$\Sigma = \{\{0, 1\}, \{2, 3, 4, 5, 5\}, \{7, 8, 9\}\}, \qquad \mathrm{T} = \{\{0, 1\}, \{2, 3, 4\}, \{4, 5, 6\}, \{6\}, \{7, 8, 9\}\},$$
$$\Pi = \{\{1, 3, 5, 7, 9\}, \{0, 2, 4, 6, 8\}\}, \qquad \Upsilon = \{\{0, 1\}, \{2, 3, 4\}, \{5, 6, 7, 8, 9\}, \{4, 3, 2\}\},$$
$$\Gamma = \{\emptyset, \{0, 2\}, \{1, 3, 5\}, \{4, 6, 8\}, \{7, 9\}\}, \qquad \Delta = \{\{0, 1\}, \{2, 3, 4\}, \{5, 6, 7, 8, 9, 10\}\}.$$

5. *You are not required to justify your answer in this question.*
   Let $A = \{0, 1, 2, 3, 4, 5\}$, and Ω be the partition of A given by $\Omega = \{\{0\}, \{1, 3, 5\}, \{2, 4\}\}$.
   Write down all the elements of the graph E_Ω of the equivalence relation R_Ω in A induced by the partition Ω.

6. Define the relation $S = (\mathbb{C}, \mathbb{C}, F)$ in $\mathbb{C}$ by $F = \{(\zeta, \xi) \in \mathbb{C}^2 : \zeta^2 - \xi^2 = ai \text{ for some } a \in \mathbb{R}\}$.
   (a) Verify that S is reflexive.
   (b) Verify that S is symmetric.
   (c) Verify that S is an equivalence relation in $\mathbb{C}$.

7. Write $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$. Define the relation $R = (\mathbb{C}^*, \mathbb{C}^*, E)$ in $\mathbb{C}^*$ by $E = \left\{(\zeta, \eta) \in (\mathbb{C}^*)^2 : \dfrac{\eta}{\zeta} \in \mathbb{R}^*\right\}$.

   (a) Verify that R is reflexive.

   (b) Verify that R is transitive.

   (c) Verify that R is an equivalence relation in $\mathbb{C}^*$.

8. Define $J = \left\{((s,t),(u,v)) \;\middle|\; \begin{array}{l} s,t,u,v \in \mathbb{N}, \text{ and} \\ [s < u \text{ or } (s = u \text{ and } t \leq v)] \end{array}\right\}$, and $T = (\mathbb{N}^2, \mathbb{N}^2, J)$. Note that $J \subset (\mathbb{N}^2)^2$.

   (a) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement '*T is reflexive*'. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   > We verify the statement ' ____(I)____ '.
   > ____(II)____ $p \in \mathbb{N}^2$. ____(III)____ $s, t \in \mathbb{N}$ ____(IV)____ .
   > We have $s = s$ ____(V)____ .
   > Then by the definition of J, we have ____(VI)____ .
   > It follows that T is reflexive.

   (b) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement '*T is anti-symmetric*'. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   > We verify the statement ' ____(I)____ '.
   > Pick any $p, q \in \mathbb{N}^2$. ____(II)____ .
   > There exist some $s, t, u, v \in \mathbb{N}$ such that $p = (s,t)$ and $q = (u,v)$.
   > Since $(p,q) \in J$, we have $(s < u$ or $(s = u$ and $t \leq v))$.
   > Then $s \leq u$ ____(III)____ $(s < u$ ____(IV)____ $t \leq v)$, by the Law of Distribution for conjunction and disjunction.
   > In particular, $s \leq u$.
   > Since ____(V)____ , we also deduce ____(VI)____ by modifying the argument above.
   > Then $s \leq u$ and ____(VII)____ . Therefore $s = u$.
   > Now recall that $(s < u$ or $(s = u$ and $t \leq v))$.
   > Then $s = u$ and ____(VIII)____ . In particular, $t \leq v$.
   > Similarly, we deduce that ____(IX)____ .
   > Then $t \leq v$ ____(X)____ $v \leq t$. Therefore ____(XI)____ .
   > Then $s = u$ and $t = v$. Hence ____(XII)____
   > It follows that T is anti-symmetric.

   (c) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement '*T is transitive*'. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

We verify the statement '_____(I)_____'.

Pick any $p, q, r \in \mathbb{N}^2$. Suppose $(p, q) \in J$ and $(q, r) \in J$.

- (Case 1.) _____(II)_____ . Then $(p, r) = (q, r) \in J$.
- (Case 2.) Suppose $q = r$. Then _____(III)_____ .
- (Case 3.) Suppose _____(IV)_____ .

  _____(V)_____ .

  Since _____(VI)_____ , we have $s < u$ or ($s = u$ and $t \leq v$).

  Since $p \neq q$, '$s = u$ and $t = v$' is _____(VII)_____ .

  Then we have $s < u$ _____(VIII)_____ ($s = u$ _____(IX)_____ $t < v$).

  Since _____(X)_____ , we can also deduce that _____(XI)_____ , by modifying the argument above.

  Now we have [$s < u$ or ($s = u$ and $t < v$)] and [$u < w$ or ($u = w$ and $v < x$)].

  - (Case 3a.) Suppose $s < u$ and $u < w$. Then _____(XII)_____ . Therefore $s < w$ or ($s = w$ and $t < x$).
  - (Case 3b.) _____(XIII)_____ . Then $s < u = w$. Therefore $s < w$ or ($s = w$ and $t < x$).
  - (Case 3c.) _____(XIV)_____ . Then $s = u < w$. Therefore $s < w$ or ($s = w$ and $t < x$).
  - (Case 3d.) _____(XV)_____ . Then $s = u = w$ and $t < v < x$. Therefore $s < w$ or ($s = w$ and $t < x$).

  Then in any case $s < w$ or ($s = w$ and $t < x$). Hence _____(XVI)_____ .

Hence in any case, $(p, r) \in J$.
It follows that T is transitive.

(d) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement '*T is strongly connected*'. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

We verify the statement '_____(I)_____'.

_____(II)_____ .

There exist some $s, t, u, v \in \mathbb{N}$ such that $p = (s, t)$ and $q = (u, v)$.
We have $s < u$ or $s = u$ or $s > u$.

- (Case 1.) Suppose $s < u$. Then _____(III)_____ . Therefore $(p, q) \in J$.
- (Case 2.) _____(IV)_____ . We have $t \leq v$ _____(V)_____ $v \leq t$.
  - (Case 2a.) _____(VI)_____ . Then $s = u$ and $t \leq v$. Therefore $s < u$ or ($s = u$ and $t \leq v$). Hence $(p, q) \in J$.
  - (Case 2b.) _____(VII)_____ . Then $u = s$ and $v \leq t$. Therefore $u < s$ or ($u = s$ and $v \leq t$). Hence _____(VIII)_____ .
- (Case 3.) _____(IX)_____ . Then _____(X)_____ . Therefore _____(XI)_____ .

Hence in any case, _____(XII)_____ .
It follows that T is strongly connected.

9. Define the relation $R = (\mathbb{R}, \mathbb{R}, P)$ in $\mathbb{R}$ by $P = \{(x, y) \in \mathbb{R}^2 : \text{There exists some } n \in \mathbb{N} \text{ such that } y = 2^n x\}$.

(a) Verify that R is a partial ordering in $\mathbb{R}$.
(b) Is R a total ordering in $\mathbb{R}$? Why?
(c) Is R an equivalence relation in $\mathbb{R}$? Why?

10. Let $R = (\mathbb{R}, \mathbb{R}, G)$ be the relation in $\mathbb{R}$ defined by $G = \{(x, y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R} \text{ and } |y - x| \leq 1\}$.

    (a) Verify that R is reflexive.
    (b) Is R symmetric? Justify your answer.
    (c) Is R anti-symmetric? Justify your answer.
    (d) Is R transitive? Justify your answer.
    (e) Is R an equivalence relation in $\mathbb{R}$? Is R a partial ordering in $\mathbb{R}$? Or neither? Why?

11. Let $A = \{\varphi \mid \varphi : \mathbb{N} \longrightarrow \mathbb{R} \text{ is a function and } \varphi(0) = 0.\ \}$. Define the relation $R = (A, A, H)$ in A by $H = \big\{(\varphi, \psi) \in A^2 : \varphi(n+1) - \varphi(n) \leq \psi(n+1) - \psi(n) \text{ for any } n \in \mathbb{N}\big\}$.

    (a) Verify that R is reflexive.
    (b) Verify that R is transitive.
    (c) Is R a partial ordering in A? Justify your answer.
    (d) Is R an equivalence relation in A? Justify your answer.

12. Define the relation $R = (\mathbb{Z}, \mathbb{Z}, G)$ in $\mathbb{Z}$ by $G = \big\{(x, y) \in \mathbb{Z}^2 : \text{There exist some } m, n \in \mathbb{N} \text{ such that } y = mx + n.\big\}$.

    (a) Verify that R is reflexive.
    (b) Verify that R is transitive.
    (c) Verify that R is not a partial ordering in $\mathbb{Z}$.
    (d) Is R is an equivalence relation in $\mathbb{Z}$? Justify your answer.

13. Let $J = [0, 1)$, $K = (0, 1]$.

Define $H = \{(x, y) \mid x \in J \text{ and } y \in K \text{ and } x + y = 1\}$, and $h = (J, K, H)$. Note that $H \subset J \times K$.

Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement '*J is of cardinality equal to K*'. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

We verify the statement (E): 'for any $x \in J$, ____(I)____ '.

Pick any ____(II)____ . Take ____(III)____ . By definition, we have $x + y = x + (1 - x) = 1$.

Since ____(IV)____ , we have $0 \leq x < 1$.

Since $x \geq 0$, we have ____(V)____ .

Since $x < 1$, we have ____(VI)____ .

Then $0 < y \leq 1$. Therefore ____(VII)____ .

Hence ____(VIII)____ .

We verify the statement (U): 'for any $x \in J$, for any $y, z \in K$, ____(IX)____ '.

____(X)____ . Suppose ____(XI)____ .

____(XII)____ , we have $x + y = 1$. Then $y = 1 - x$.

Since $(x, z) \in H$, we have ____(XIII)____ . Then ____(XIV)____ .

Therefore ____(XV)____ .

We verify the statement (S): ' ____(XVI)____ such that ____(XVII)____ '.

____(XVIII)____ . Take $x = 1 - y$. By definition, we have $x + y = (1 - y) + y = 1$.

Since $y \in K$, we have ____(XIX)____ .

Since ____(XX)____ , we have $x = 1 - y < 1 - 0 = 1$.

Since $y \leq 1$, we have ____(XXI)____ .

Then $0 \leq x < 1$. Therefore $x \in J$.

Hence ____(XXII)____ .

We verify the statement (I): ' ____(XXIII)____ if $(x, y) \in H$ and $(w, y) \in H$ ____(XXIV)____ '.

Pick any $x, w \in J$, $y \in K$. ____(XXV)____ .

____(XXVI)____ , we have $x + y = 1$. Then $x = 1 - y$.

Since $(w, y) \in H$, we have ____(XXVII)____ . Then $w = 1 - y$.

Therefore ____(XXVIII)____ .

By (E), (U), ____(XXIX)____ . By (S), ____(XXX)____ . By (I), ____(XXXI)____ .

Hence h is a bijective function from J to K with graph H.

It follows that ____(XXXII)____ .

14. Let $J = [0, 1)$, $L = (0, 1)$, $M = [0, +\infty)$, $N = (0, +\infty)$.

    (a) By writing down appropriate bijective functions, verify that $J \sim M$ and $L \sim N$.

    (b) By writing down an appropriate bijective function, verify that $L \sim \mathbb{R}$.

15. Let $D = \{z \in \mathbb{C} : |z| < 1\}$, $H = \{w \in \mathbb{C} : \mathsf{Im}(w) > 0\}$.

Define $F = \left\{ (z, w) \,\middle|\, z \in D \text{ and } w \in H \text{ and } w = \dfrac{z + i}{iz + 1} \right\}$, and $f = (D, H, F)$. Note that $F \subset D \times H$.

(a) Is f a function? Justify your answer.
(b) Is it true that $D \sim H$? Justify your answer.

16. Let $I = (0, +\infty)$, $J = [-1, 1]$.

    (a) Prove that $\dfrac{1}{a+1} \in J$ for any $a \in I$.

    (b) Define the function $g : I \longrightarrow J$ by $g(x) = \dfrac{1}{x+1}$ for any $x \in I$. Is g injective? Justify your answer.

    (c) Apply the Schröder-Bernstein Theorem to prove that $I \sim J$.

17. Let $A = [-1, 1]$, $B = (-4, -2] \cup [2, 4)$.

    (a) Name one injective function from A to B, if there is any at all, and verify that it is indeed an injective function from A to B.
    (b) Apply the Schröder-Bernstein Theorem, or otherwise, to prove that A is of cardinality equal to B.

18.$^\diamondsuit$ Let $A = [1010, 1050] \backslash \{1030\}$ and $B = (2040, 2050) \cup ([2060, +\infty) \cap \mathbb{Q})$.

    (a) Name one injective function from A to B, if there is any at all, and verify that it is indeed an injective function from A to B.
    (b) Apply the Schröder-Bernstein Theorem to prove that $A \sim B$.

19. Let A, B, C be sets. Prove the statements below:

    (a) $A \sim A$.
    (b) *Suppose $A \sim B$. Then $B \sim A$.*
    (c) *Suppose $A \sim B$ and $B \sim C$. Then $A \sim C$.*
    (d) $A \lesssim A$.
    (e) *Suppose $A \lesssim B$ and $B \lesssim C$. Then $A \lesssim C$.*

20. (a) Let A, B, C, D be sets, and $f : A \longrightarrow C$, $g : B \longrightarrow D$ be functions. Define the function $f \times g : A \times B \longrightarrow C \times D$ by $(f \times g)(x, y) = (f(x), g(y))$ for any $x \in A$, for any $y \in B$.
        i. Suppose f, g are surjective. Verify that $f \times g$ is surjective.
        ii. Suppose f, g are injective. Verify that $f \times g$ is injective.
        iii. Suppose f, g are bijective. Verify that $f \times g$ is bijective.

    (b) Let A, B, C, D be sets. Suppose $A \sim C$ and $B \sim D$. Prove that $A \times B \sim C \times D$.
    Remark. Hence the statement below holds:
    Let A, B be sets. Suppose $A \sim B$. Then $A^2 \sim B^2$.

21. Let A be a non-empty set. Define the function $\chi : \mathfrak{P}(A) \longrightarrow \mathsf{Map}(A, \{0, 1\})$ by $\chi(S) = \chi_S^A$ for any $S \in \mathfrak{P}(A)$.
    Here, for any subset S of A, $\chi_S^A : A \longrightarrow \{0, 1\}$ is the characteristic function of S in A, defined by

$$\chi_S^A(x) = \begin{cases} 1 & \text{if} \quad x \in S, \\ 0 & \text{if} \quad x \in A \backslash S. \end{cases}$$

    (a)$^\clubsuit$ Verify that χ is surjective.
    (b)$^\clubsuit$ Verify that χ is injective.
    (c) Is it true that $\mathfrak{P}(A) \sim \mathsf{Map}(A, \{0, 1\})$? Justify your answer.

22. Define the functions $\sigma, \tau : \mathbb{N} \longrightarrow \mathbb{N}$ by $\sigma(n) = 2n$, $\tau(n) = 2n + 1$ for any $n \in \mathbb{N}$.
    Let B be a non-empty set. Define the function $f : \mathsf{Map}(\mathbb{N}, B) \longrightarrow (\mathsf{Map}(\mathbb{N}, B))^2$ by $f(\varphi) = (\varphi \circ \sigma, \varphi \circ \tau)$ for any $\varphi \in \mathsf{Map}(\mathbb{N}, B)$.

    (a)$^\clubsuit$ Verify that f is surjective.
    (b)$^\clubsuit$ Verify that f is injective.
    (c) Is it true that $\mathsf{Map}(\mathbb{N}, B) \sim (\mathsf{Map}(\mathbb{N}, B))^2$? Justify your answer.

23.$^\clubsuit$ *In this question, we are going to give another proof for Cantor's Theorem on the power set of any given set.*

    (a) Let A be a set, and $f : A \longrightarrow \mathfrak{P}(A)$ be a function. Define $C_f = \{x \in A : x \notin f(x)\}$. (Note that $C_f \in \mathfrak{P}(A)$.)
        i. Dis-prove the statement '*there exists some $z \in A$ such that $f(z) = C_f$.*
        ii. Hence deduce that f is not surjective.

    Remark. The set C_f is called **Cantor's diagonal set for the function f.**

    (b) Apply the above results to prove Cantor's Theorem on the power set of any given set.