MATH1050 Exercise 9 Supplement (Answers)

1. ——
2. (a) ——
   (b) ——
   (c) Yes.
3. ——
4. (a) Yes.
   (b) No.
5. (a) No.
   (b) No.
6. (a) No.
   (b) Yes.
7. (a) No. Note that $f(0) = f(1)$.
   (b) No. Note that $f(x) \neq 1$ for any $x \in \mathbb{R}$.
8. (a) i. No. Note that $f(2) = f(\sqrt[3]{2})$.
   ii. Yes.
   (b) i. Yes.
   ii. No. Note that $f(x) = 2$ for any $x \in \mathbb{R}$.
9. (a) ——
   (b) i. *Hint.* It turns out that for any $x \in B$, $g(x) \neq 2$. (Why?)
   ii. Yes.
10. (a) ——
    (b) i. ——
    ii. ——
    iii. Yes.
11. (a) i. No.
    ii. No.
    (b) ——
    (c) i. Yes.
    ii. Yes.
    iii. Yes. The 'formula of definition' for the inverse function $g^{-1} : (-1, +\infty) \longrightarrow (1+\infty)$ of the function g is given by $g^{-1}(y) = 1+\sqrt{y+1}$ for any $y \in (-1, +\infty)$.
12. (a) ——
    (b) ——
    (c) The 'formula of definition' of the function $g^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$ is given by $g^{-1}(y) = \log_{10}(y + \sqrt{1+y^2})$ for any $y \in \mathbb{R}$.
13. (a) ——
    (b) i.
    ii.
    iii. Yes. The 'formula of definition' for the inverse function $g^{-1} : I \longrightarrow \mathbb{R}$ of the function g is given by $g^{-1}(y) = \ln\left(\dfrac{y^2}{1-y^2}\right)$ for any $y \in I$.
14. (a) ——
    i. Yes.
    ii. Yes.
    iii. Yes. The 'formula of definition' for the inverse function $f^{-1} : \mathbb{R}\backslash\{3\} \longrightarrow \mathbb{R}\backslash\{2\}$ of the function f is given by $f^{-1}(y) = \dfrac{2y}{y-3}$ for any $y \in \mathbb{R}\backslash\{3\}$.
15. (a) Yes.
    (b) Yes.
    (c) Yes. The 'formula of definition' of the inverse function $f^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$ of the function f is given by

$$f^{-1}(y) = \begin{cases} \sqrt{y} & \text{if } y \geq 0 \\ -\sqrt{|y|} & \text{if } y < 0 \end{cases}$$

16. (a) Yes.
    (b) Yes.
    (c) Yes. The 'formula of definition' for the inverse function $f^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$ of the function f is given by $f^{-1}(y) = y^{\frac{2}{3}}$ for any $y \in \mathbb{R}$.
17. (a) ——
    (b) i. Yes.
    ii. Yes.
    iii. Yes. The 'formula of definition' for the inverse function $f^{-1} : (-1, 1) \longrightarrow \mathbb{R}$ of the function f is given by

$$f^{-1}(y) = \begin{cases} \sqrt{\dfrac{y}{1-y}} & \text{if } y > 0 \\ -\sqrt{\dfrac{-y}{1+y}} & \text{if } y < 0 \end{cases}.$$

18. ——

    Hint. Find out how to 'complete the square' of a quadratic expression if you do not know how to proceed.
19. (a) ——
    (b) No.
    (c) Yes.
20. (a) ——
    (b) ——
    (c) i. ——
    ii. No.
    (d) i. ——
    ii. No.
21. ——
22. (a) ——
    (b) i. ——
    ii. ——
    iii. The inverse function of f, namely, the function $f^{-1} : \mathbb{C}\backslash\{1\} \longrightarrow \mathbb{C}\backslash\{-2\}$, is given by $f^{-1}(w) = \dfrac{2w-i}{-w+1}$ for any $w \in \mathbb{C}\backslash\{1\}$.
23. (a) ——
    (b) ——

(c) The inverse function $f_{a,b}$ of is the function $f_{a',b'}$, given explicitly by $f_{a',b'}(w) = \frac{\bar{a}}{|a|^2-|b|^2}w + \frac{-b}{|a|^2-|b|^2}\bar{w}$ for any $w \in \mathbb{C}$.

(d) f is not bijective.

24. (a) $\mathsf{Re}(f(z)) = (\mathsf{Re}(z))^3$ and $\mathsf{Im}(f(z)) = (\mathsf{Re}(z)^2 + 1)\mathsf{Im}(z)$.

(b) ——

(c) ——

(d) The inverse function $f^{-1} : \mathbb{C} \longrightarrow \mathbb{C}$ of the function f is given by $f^{-1}(\zeta) = \sqrt[3]{\mathsf{Re}(\zeta)} + \frac{i\mathsf{Im}(\zeta)}{(\sqrt[3]{\mathsf{Re}(\zeta)})^2+1}$ for any $\zeta \in \mathbb{C}$.

25. (a) ——

(b) ——

(c) $(h \circ h)(z) = -z$ for any $z \in \mathbb{C}$.

(d) Yes.

26. (a) i. No.
ii. No.

(b) i. Yes.
ii. Yes.

27. ——

28. ——

29. ——

30. (a) True.

(b) False.

31. (a) i. Yes.
ii. No.

(b) ——

(c) i. No.
ii. Yes.
iii. No.
iv. No.

32. ——

33. ——

34. (a) ——

(b) i. $\kappa = \sqrt[3]{3}, \lambda = \sqrt[3]{9}$.
ii. $-\kappa - \lambda$, $-\kappa\omega - \lambda\omega^2$, $-\kappa\omega^2 - \lambda\omega$.

(c) i. $g(y) = x^3 - 15x + 10\sqrt{5}$.
ii. $-1 - 2\sqrt{5}$ and $-1 - \sqrt{5}$ (repeated).

35. (a) ——

Hint. The equality $x^3 + \sigma^3 + \tau^3 - 3\sigma\tau x = [x + (\sigma + \tau)][x + (\sigma\omega + \tau\omega^2)][x + (\sigma\omega^2 + \tau(\omega))]$ holds for polynomials.

Remark. In fact, each of $-(\sigma + \tau)$, $-(\sigma\omega + \tau\omega^2)$, $-(\sigma\omega^2 + \tau\omega)$ is a root of $f(x)$ in $\mathbb{C}$. According to the factorization of $f(x)$, they are all the roots of $f(x)$ in $\mathbb{C}$. (Why?)

(b) i. $g(y) = (y-\alpha)^3 + s(y-\alpha) + t$ as polynomials, in which $\alpha = -\frac{p}{3}$, $s = q - \frac{p^3}{3}$, $t = \frac{2p^3}{27} - \frac{pq}{3} + r$.
ii. ——

(c) ——

Hint. For each $\gamma \in \mathbb{C}$, define the function $F_\gamma : \mathbb{C} \longrightarrow \mathbb{C}$ by $F_\gamma(z) = \frac{G(z)-\gamma}{A}$ for any $z \in \mathbb{C}$. Prove that F_γ has a root in $\mathbb{C}$.

(d) ——

36. ——

37. (a) It is the empty set.

(b) 5.

(c) 5.

(d) $5, 7$.

(e) $5, 7, 9$.

(f) $5, 7, 9$.

(g) It is the empty set.

(h) $0, 1$.

(i) It is the empty set.

(j) $0, 1$.

(k) $0, 1, 2$.

(l) $0, 1, 2, 3, 4$.

38. (a) $\alpha = 0$.

(b) $\beta = 1$, $\gamma = 2$.

(c) $\delta = -1$, $\varepsilon = 0$, $\zeta = 0.25$.

(d) $\mu = -1$, $\nu = 0.5$, $\sigma = 1$, $\tau = 2$.

39. (a) $\alpha = -2$, $\beta = 2$, $\gamma = 0$, $\delta = -3$.

(b) $\varepsilon = -0.5$, $\zeta = -1$, $\eta = 0$, $\theta = 0.05$.

40. (a) $\alpha = 1$.

(b) $\beta = 0$.

(c) $\gamma = 0$, $\delta = 1$.

(d) $\varepsilon = 1$.

(e) $\zeta = 1$.

(f) $\eta = 0$, $\theta = 1$.

(g) $\kappa = 0$, $\lambda = 1$, $\mu = 2$.

(h) $\nu = 0$, $\xi = 1$, $\rho = 2$.

(i) $\sigma = 1$, $\tau = 3$.

(j) $\varphi = 1$, $\psi = 3$.

41. (a) 2

(b) -5

(c) $\alpha = 1/5$, $\beta = 2$, $\gamma = 14$, $\delta = 18$.

42. (a) 0.

(b) $1, 3$.

(c) 1.

(d) $\alpha = -\sqrt{3}$, $\beta = -1$, $\gamma = 3/2$, $\delta = 2$, $\varepsilon = 5/2$, $\zeta = 8/3$.

43. (a) i. $f([1,2]) = \left[\frac{1}{4}, 1\right]$.

ii. $f((0,1)) = (1,+\infty)$.

iii. $f^{-1}([1,4]) = \left[-1,-\frac{1}{2}\right] \cup \left[\frac{1}{2},1\right]$.

iv. $f^{-1}([-1,1]) = (-\infty,-1] \cup [1,+\infty)$.

(b) ——

44. (a) i. $\lim_{t\to-\infty} f(t) = 0$, and $\lim_{t\to+\infty} f(t) = 0$.

ii. f is strictly decreasing on $(-\infty,-2]$.
f is strictly increasing on $[-2,2]$.
f is strictly decreasing on $[2,+\infty)$.

iii. f attains a relative minimum at -2, with $f(-2) = -2$.
f attains a relative maximum at 2, with $f(2) = 2$.

iv. f attains the absolute minimum at -2.
f attains the absolute maximum at 2.

(b) i. A. $f(\mathbb{R}) = [-2,2]$.

B. $f([0,1]) = \left[0,\frac{8}{5}\right]$.

C. $f([1,3]) = \left[\frac{8}{5},2\right]$.

D. $f([1,+\infty)) = (0,2]$.

ii. ——

(c) i. A. $f^{-1}(\mathbb{R}) = \mathbb{R}$.

B. $f^{-1}([0,2]) = [0,+\infty)$.

C. $f^{-1}((1,3)) = (4-2\sqrt{3},4+2\sqrt{3})$.

D. $f^{-1}((0,1)) = (-\infty,-4-2\sqrt{3}) \cup (-4+2\sqrt{3},4-2\sqrt{3}) \cup (4+2\sqrt{3},+\infty)$

ii. ——

45. ——

46. ——

47. ——

48. (a) i. ——

ii. The function $f\circ\gamma : \mathbb{R} \longrightarrow \mathbb{C}$ is given by $(f\circ\gamma)(t)(1-t^2)+(2t)i$ for any $t\in\mathbb{R}$.

iii. ——

(b) ——

49. ——

50. ——

51. ——

52. ——

53. (a) No.
(b) No.
(c) No.
(d) No.

54. (a) Yes.
(b) No.
(c) No.

55. (a) ——
(b) No.
(c) No.

56. (a) ——
(b) No.
(c) No.

57. (a) No.
(b) No.
(c) Yes.

58. (a) ——
(b) No.

59. (a) i. (A,A,E_α) is a function iff $\alpha \le 1$.

ii. (A,A,E_α) is an injective function iff $\alpha \le 1$.

iii. (A,A,E_α) is a surjective function iff $\alpha = 1$.

iv. (A,A,E_α) is a bijective function iff $\alpha = 1$.

(b) i. (A,A,F_β) is a function iff $-1 \le \beta < 1$.

ii. (A,A,F_β) is an injective function iff $-1 \le \beta < 1$.

iii. (A,A,F_β) is a surjective function iff $\beta = -1$.

iv. (A,A,F_β) is a bijective function iff $\beta = -1$.

60. (a) ——

(b) The function $f : A \longrightarrow B$ is given by $f(x) = x^{\frac{4}{3}}$ for any $x\in A$.

(c) f is not injective.

(d) f is surjective.

61.$^\diamond$

(a) No.
(b) Yes.

62. ——

63. (a) ——

(b) The function $f : A \longrightarrow A$ is given by $f(x) = \dfrac{x}{x-1}$ for any $x\in A$.

(c) ——

(d) The inverse function $f^{-1} : A \longrightarrow A$ is given by $f^{-1}(y) = \dfrac{y}{y-1}$ for any $y\in A$.

(e) The function $f\circ f : A \longrightarrow A$ is given by $(f\circ f)(x) = x$ for any $x\in A$.
The function $f\circ f\circ f : A \longrightarrow A$ is given by $(f\circ f\circ f)(x) = f(x) = \dfrac{x}{x-1}$ for any $x\in A$.

64. ——

65. ——

66. ——

67. ——

68. ——

69. ——

70. ——

71. ——

72. ——

73. ——

74. (a) False.

(b) False.
(c) False.
(d) False.
(e) False.
(f) False.

75. ——

76. ——

77. ——

78. (a) i. ——
ii. ——
iii. ——
iv. ——
v. ——
vi. ——
vii. ——
viii. $g(\ell) \neq 0$.
ix. g is not continuous at ℓ.

(b) ——-

(c) ——-

79. (a) ——

(b) i. $f(0) = 1$.
ii. ——

(c) ——

80. ——

81. ——

82. ——