MATH1050 Exercise 9 (Answers and selected solution)

1. **Solution.**

   (a) M is formally formulated as: 'For any set A, for any functions $f, g : A \longrightarrow A$, the equality $g \circ f = f \circ g$ as functions holds.'

   Hence $\sim M$ reads: 'There exist some set A, some functions $f, g : A \longrightarrow A$ such that $g \circ f \neq f \circ g$ as functions.'

   (b) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions defined by $f(x) = \dfrac{x^2}{1+x^2}$, $g(x) = x - 1$ for any $x \in \mathbb{R}$.

   i. For any $x \in \mathbb{R}$, we have

   $$\begin{aligned}(g \circ f)(x) = g(f(x)) &= g\left(\frac{x^2}{1+x^2}\right) = \frac{x^2}{1+x^2} - 1 = -\frac{1}{1+x^2},\\ (f \circ g)(x) = f(g(x)) &= f(x-1) = \frac{(x-1)^2}{1+(x-1)^2}.\end{aligned}$$

   ii. We have $(g \circ f)(0) = -1$ and $(f \circ g)(0) = \dfrac{1}{2}$. Hence $(g \circ f)(0) \neq (f \circ g)(0)$.

   iii. There exists some $x_0 \in \mathbb{R}$, namely, $x_0 = 0$, such that $(g \circ f)(x_0) \neq (f \circ g)(x_0)$. Hence it is not true that $g \circ f = f \circ g$ as functions.

   (c) Let $A = \{0, 1\}$.

   Define $f, g : A \longrightarrow A$ by $f(0) = f(1) = 0$, $g(0) = g(1) = 1$.

   $g \circ f : A \longrightarrow A$ is given by $(g \circ f)(0) = g(f(0)) = g(0) = 1$, $(g \circ f)(1) = g(f(1)) = g(0) = 1$.

   $g \circ f : A \longrightarrow A$ is given by $(f \circ g)(0) = f(g(0)) = f(1) = 0$, $(f \circ g)(1) = f(g(1)) = f(1) = 0$.

   Take $x_0 = 0$. We have $(g \circ f)(x_0) = 1$ and $(f \circ g)(x_0) = 0$. Then $(g \circ f)(x_0) \neq (f \circ g)(x_0)$.

   Therefore $g \circ f \neq f \circ g$ as functions.

   Remark. Let A be a set. (This set is fixed in our subsequent discussion.) Suppose $f, g : A \longrightarrow A$ are two functions from the set A to A itself.

   - When we want to verify that $g \circ f$, $f \circ g$ are the same function from A to A, we have to verify that for any $x \in A$, $(g \circ f)(x) = (f \circ g)(x)$.
   - To verify that $g \circ f$, $f \circ g$ are not the same function from A to A, we check that there exists some $x_0 \in A$ such that $(g \circ f)(x_0) \neq (f \circ g)(x_0)$. Hence we have to name an appropriate x_0 and show that $(g \circ f)(x_0) \neq (f \circ g)(x_0)$.

2. **Answer.**

   (a) (I) Pick any
   Alternative answer. Let
   Alternative answer. Suppose
   Alternative answer. Assume
   Alternative answer. Take any

   (II) $x = (y+1)^{\frac{3}{5}}$

   (III) $f(x) = x^{\frac{5}{3}} - 1 = \left[(y+1)^{\frac{3}{5}}\right]^{\frac{5}{3}} - 1 = (y+1) - 1 = y$

   (IV) f is surjective.

   (b) (I) $x, w \in \mathbb{R}$

   (II) Suppose
   Alternative answer. Assume

   (III) $f(x) + 1 = f(w) + 1$

   (IV) $(w^{\frac{5}{3}})^{\frac{3}{5}} = w$

   (V) f is injective

3. **Answer.**

(a) (I) Take
Alternative answer. Let
Alternative answer. Define
Alternative answer. Pick
Alternative answer. Suppose
Alternative answer. Assume
(II) for any $x \in \mathbb{R}$
(III) Suppose
Alternative answer. Assume
(IV) there existed some $x_0 \in \mathbb{R}$ such that $f(x_0) = y_0$.
(V) 0
(VI) $\left(x_0 - \frac{1}{2}\right)^2 + \frac{3}{4} \geq 0 + \frac{3}{4}$
(VII) f is not surjective.

(b) (I) $w_0 = 2$.
(II) $x_0 \neq w_0$.
(III) $f(w_0) = \frac{2}{2^2+1} = \frac{2}{5}$
(IV) $f(w_0)$
(V) f is not injective

4. ——

5. **Answer.**

(a) No. Note that $f(0) = f(1)$.

(b) No. Note that $f(x) \neq 1$ for any $x \in \mathbb{R}$.

6. **Solution.**

(a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^4 - 4x^2$ for any $x \in \mathbb{R}$.

i. We verify that f is not injective:
- Take $x_0 = 0$, $w_0 = 2$. We have $x_0, w_0 \in \mathbb{R}$ and $x_0 \neq w_0$. Also, $f(x_0) = 0 = f(w_0)$.

ii. We verify that f is not surjective:
- Take $y_0 = -5$.
  Pick any $x \in \mathbb{R}$. We have $f(x) = x^4 - 4x^2 = (x^2-2)^2 - 4 \geq -4 > -5$. Then $f(x) \neq -5$.
  Hence, for any $x \in \mathbb{R}$, $f(x) \neq y_0$.

(b) Let $x \in (\sqrt{2}, +\infty)$. $x^4 - 4x^2 = (x^2-2)^2 - 4 > 0 - 4 = -4$.

(c) Let $g : (\sqrt{2}, +\infty) \longrightarrow (-4, +\infty)$ be the function defined by $g(x) = x^4 - 4x^2$ for any $x \in (\sqrt{2}, +\infty)$.

i. Pick any $x, w \in (\sqrt{2}, +\infty)$. Suppose $g(x) = g(w)$. Then $x^4 - 4x^2 = w^4 - 4w^2$.
Therefore $(x-w)(x+w)(x^2+w^2) = (x^2-w^2)(x^2+w^2) = 4(x^2-w^2) = 4(x-w)(x+w)$.
Then $(x-w)(x+w)(x^2+w^2-4) = 0$.
Note that $x \geq \sqrt{2} > 0$ and $w \geq \sqrt{2} > 0$. Then $x + w > 0$ and $x^2 + w^2 - 4 > 0$.
Then $x = w$.
It follows that g is injective.

ii. Pick any $y \in (-4, +\infty)$. Note that $y + 4 > 0$. Then $\sqrt{y+4}$ is well-defined and $2 + \sqrt{4+y} > 2$. Therefore $\sqrt{2+\sqrt{4+y}}$ is well-defined and $\sqrt{2+\sqrt{4+y}} > \sqrt{2}$.
Take $x = \sqrt{2+\sqrt{4+y}}$. Note that $x \in (\sqrt{2}, +\infty)$.
We have $g(x) = x^4 - 4x^2 = (\sqrt{2+\sqrt{4+y}})^4 - 4(\sqrt{2+\sqrt{4+y}})^2 == (2+\sqrt{4+y})^2 - 4(2+\sqrt{4+y}) + 4 - 4 = [(2+\sqrt{4+y}) - 2]^2 - 4 = (4+y) - 4 = y$.
It follows that g is surjective.

iii. Since g is both injective and surjective, g is bijective. Its inverse function $g^{-1} : (-4, +\infty) \longrightarrow (\sqrt{2}, +\infty)$ is given by $g^{-1}(y) = \sqrt{2+\sqrt{4+y}}$ for any $y \in (-4, +\infty)$.

Remark. Although f and g have the same 'formula of definition', one is bijective and the other is not. So when talking about a function, be aware of its domain and its range, and don't just look at its 'formula of definition'.

7. **Answer.**
   (a) $J = (1, +\infty)$.
   (b) $f^{-1}(y) = \frac{1}{4}\left(\ln\left(\frac{y+1}{y-1}\right)\right)^2$ for any $y \in J$.

8. **Answer.**
   (a) ——
   (b) No.

9. **Answer.**
   (a) ——
   (b) The inverse function $f^{-1} : \mathbb{C} \longrightarrow \mathbb{C}$ of the function f is given by $f^{-1}(z) = \bar{z}$ for any $z \in \mathbb{C}$.
   Comment. The conjugate of the conjugate of a complex number is the complex number itself.

10. **Answer.**
    (a) No.
    (b) No.

11. **Answer.**
    (a) ——
    (b) i. ——
    ii. ——
    iii. The 'formula of definition' for inverse function $f^{-1} : \mathbb{C}\backslash\{a/c\} \longrightarrow \mathbb{C}\backslash\{-d/c\}$ of the function f is given by $f^{-1}(\zeta) = \frac{d\zeta - b}{-c\zeta + a}$ for any $\zeta \in \mathbb{C}\backslash\{a/c\}$.

12. (a) *Hint.* The crucial step is to apply Triangle Inequality to obtain

$$f(b) = b^3\left(1 + \frac{p}{b} + \frac{q}{b^2} + \frac{r}{b^3}\right) \geq b^3\left(1 - \frac{|p|}{b} - \frac{|q|}{b^2} - \frac{|r|}{b^3}\right).$$

    (b) *Hint.* For each $\gamma \in \mathbb{R}$, apply the result in the previous to the function $f_\gamma : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f_\gamma(x) = \frac{g(x) - \gamma}{A}$ for any $x \in \mathbb{R}$.

13. **Answer.**
    (a) $1, 2$.
    (b) $0, 1$.
    (c) It is the empty set.
    (d) $1, 2$.
    (e) $2, 3, 4$.
    (f) 2.
    (g) $0, 1, 2, 3, 4$.
    (h) $1, 2$.
    (i) $2, 3, 4$.

14. **Answer.**
    (a) $\alpha = -1$.
    (b) $\beta = 1$, $\gamma = \frac{5}{4}$.
    (c) $\delta = -1$, $\varepsilon = \frac{5}{3}$.
    (d) $\zeta = -\sqrt{2} + 1$, $\eta = \sqrt{2} + 1$.
    (e) $\theta = -\sqrt{2} + 1$, $\kappa = -\frac{\sqrt{5}}{2} + 1$, $\lambda = \frac{\sqrt{5}}{2} + 1$, $\mu = \sqrt{2} + 1$.
    (f) $\nu = -\frac{1}{\sqrt{2}} + 1$, $\xi = \frac{1}{\sqrt{2}} + 1$.
    (g) $\rho = -\frac{1}{\sqrt{2}} + 1$, $\sigma = \frac{1}{\sqrt{2}} + 1$.
    (h) $\tau = -\sqrt{2} + 1$, $\varphi = -\frac{1}{\sqrt{2}} + 1$, $\psi = \frac{1}{\sqrt{2}} + 1$, $\omega = \sqrt{2} + 1$.

15. **Answer.**

(a) $\alpha = 0, \beta = 3, \gamma = 1, \delta = -1$.

(b) $\varepsilon = -2$, $\zeta = -0.25$, $\eta = -1$, $\theta = 0$, $\kappa = 1$.

16. **Answer.**

(a) $a = 2, b = 3$.

(b) $1, 4$.

(c) $\alpha = 0$, $\beta = 1$, $\gamma = 2$, $\delta = 4$.

17. **Answer.**

(a) (I) Suppose $y \in f(S)$

(II) there exists some $x \in S$ such that $y = f(x)$

(III) $y = f(x) = 2x^4 - 4 \geq 2 \cdot 1^4 - 4 = -2$

(IV) Since $x \leq 2$

(V) Take $x = \sqrt[4]{\dfrac{y+4}{2}}$

(VI) $x = \sqrt[4]{\dfrac{y+4}{2}} \geq 1$

(VII) Since $y \leq 28$, we have $\dfrac{y+4}{2} \leq 16$

(VIII) $x = \sqrt[4]{\dfrac{y+4}{2}} \leq 2$

(IX) $f(x) = 2x^4 - 4 = 2\left(\sqrt[4]{\dfrac{y+4}{2}}\right)^4 - 4 = 2 \cdot \dfrac{y+4}{2} - 4 = y + 4 - 4 = y$

(X) $y \in f(S)$

(b) (I) Suppose $x \in f^{-1}(U)$

(II) there exists some $y \in U$ such that $y = f(x)$

(III) $y \in U$

(IV) $2x^4 - 4 = f(x) = y \leq 4$

(V) Suppose $x \in [-\sqrt{2}, \sqrt{2}]$

(VI) Define $y = f(x)$

(VII) $y = f(x) = 2x^4 - 4 \leq 4$

(VIII) $y = f(x) = 2x^4 - 4 \geq -6$

(IX) and

(X) $x \in f^{-1}(U)$

18. **Answer.**

(a) $(p, q) = (0, 0)$ or $(p, q) = (0, 1)$. $(s, t) = (1, \tau)$, provided that $-2 \leq \tau < 2$.

(b) $(p, q) = (1, 1)$ and $(s, t) = (2, 1)$. *Alternative answer*: $(p, q) = (1, 2)$ and $(s, t) = (2, 2)$.

(c) $(m, n) = (0, 0)$. $(p, q) = (1, 1)$ or $(p, q) = (1, 2)$.