MATH1050 Exercise 9

1. (a) Consider the statement M below:

   - *Let A be a set, and $f, g : A \longrightarrow A$ be functions. The equality $g \circ f = f \circ g$ as functions holds.*

   Write down the negation $\sim M$ of the statement M.

   (b) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions defined by $f(x) = \dfrac{x^2}{1+x^2}$, $g(x) = x - 1$ for any $x \in \mathbb{R}$.

   i. Compute the respective 'formulae of definition' of the functions $g \circ f$, $f \circ g$ explicitly.

   ii. Choose some $x_0 \in \mathbb{R}$ so that $(g \circ f)(x_0) \neq (f \circ g)(x_0)$.

   iii. Is it true that $g \circ f = f \circ g$ as functions? Justify your answer.

   Remark. Hence we have dis-proved the statement M with a counter-example. (Why?)

   (c) Define $A = \{0, 1\}$. Prove that there exist some functions $f, g : A \longrightarrow A$ such that $g \circ f \neq f \circ g$ as functions.

   Remark. Hence we have dis-proved the statement M with another counter-example. (Why?)

2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^{\frac{5}{3}} - 1$ for any $x \in \mathbb{R}$.

   (a) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the surjectivity of f. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   > [We want to verify the statement '*for any $y \in \mathbb{R}$, there exists some $x \in \mathbb{R}$ such that $y = f(x)$.*']
   >
   > ____(I)____ $y \in \mathbb{R}$.
   > Take ____(II)____ .
   > Note that $x \in \mathbb{R}$.
   > Also note that ____(III)____ .
   > It follows that ____(IV)____ .

   (b) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the injectivity of f. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   > [We want to verify the statement '*for any $x, w \in \mathbb{R}$, if $f(x) = f(w)$ then $x = w$.*']
   >
   > Pick any ____(I)____ .
   > ____(II)____ $f(x) = f(w)$.
   > Then $x^{\frac{5}{3}} =$ ____(III)____ $= w^{\frac{5}{3}}$.
   > Since $x, w \in \mathbb{R}$, we have $x = (x^{\frac{5}{3}})^{\frac{3}{5}} =$ ____(IV)____ .
   > It follows that ____(V)____ .

3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{x}{x^2+1}$ for any $x \in \mathbb{R}$.

   (a) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the non-surjectivity of f. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

[We want to verify the statement '*there exists some* $y_0 \in \mathbb{R}$ *such that for any* $x \in \mathbb{R}$ *such that* $y \neq f(x)$.']

____(I)____ $y_0 = 1$.

We verify, using the method of proof-by-contradiction, that ________(II)________, $f(x) \neq y_0$:

* ____(III)____ it were true that __________(IV)__________.

  Then $\dfrac{x_0}{{x_0}^2+1} = f(x_0) = y_0 = 1$.

  Therefore ___(V)___ $= {x_0}^2 - x_0 + 1 =$ ________(VI)________ > 0. Contradiction arises.

It follows that ________(VII)________.

(b) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the non-injectivity of f. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

[We want to verify the statement '*there exists some* $x_0, w_0 \in \mathbb{R}$, *such that* $f(x_0) = f(w_0)$ *and* $x \neq w$.']

Take $x_0 = \dfrac{1}{2}$, ____(I)____.

Note that $x_0, w_0 \in \mathbb{R}$.

Also note that ________(II)________.

We have $f(x_0) = \dfrac{1/2}{(1/2)^2+1} = \dfrac{2}{5}$ and __________(III)__________.

Then $f(x_0) =$ ____(IV)____.

It follows that ________(V)________.

4. Let $f : (0, +\infty) \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{x^2 - 2x + 4}{x^2 + 2x + 4} \cos\left(\dfrac{1}{x\sqrt{x}}\right)$ for any $x \in (0, +\infty)$.

   (a) Verify that f is not injective.

   (b) i. Verify that $\left|\dfrac{x^2 - 2x + 4}{x^2 + 2x + 4}\right| \leq 1$ for any $x \in (0, +\infty)$.

   Remark. A very simple answer can be obtained without using calculus.

   ii. Apply the previous part, or otherwise, to verify that f is not surjective.

5.[◇] Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{Q} \\ x^3 & \text{if} \quad x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$.

   (a) Is f injective? Justify your answer.

   (b) Is f surjective? Justify your answer.

6. (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^4 - 4x^2$ for any $x \in \mathbb{R}$.

   i. Is f injective? Justify your answer.

   ii. Is f surjective? Justify your answer.

   (b) Verify that for any $x \in (\sqrt{2}, +\infty)$, $x^4 - 4x^2 > -4$.

   (c) Let $g : (\sqrt{2}, +\infty) \longrightarrow (-4, +\infty)$ be the function defined by $g(x) = x^4 - 4x^2$ for any $x \in (\sqrt{2}, +\infty)$.

   i. Is g injective? Justify your answer.

   ii. Is g surjective? Justify your answer.

   iii. Is g bijective? If *yes*, also write down the 'formula of definition' for its inverse function.

7. *You are not required to prove your answers in this question.*

   The function $f : (0, +\infty) \longrightarrow J$, given by $f(x) = \dfrac{e^{\sqrt{x}} + e^{-\sqrt{x}}}{e^{\sqrt{x}} - e^{-\sqrt{x}}}$ for any $x \in (0, +\infty)$ is known to be a bijective function from $(0, +\infty)$ to the set J.

   (a) Express the set J explicitly as an interval.

   (b) Write down the explicit 'formula of definition' for the inverse function f^{-1} of the function f.

8. Let $n \in \mathbb{N}\backslash\{0, 1\}$, and $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = z^n$ for any $z \in \mathbb{C}$.

   (a) Verify that f is surjective.

   (b) Is f injective? Why?

9. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = \bar{z}$ for any $z \in \mathbb{C}$.

   (a) Verify that f is bijective.

   (b) Write down the 'formula of definition' of the inverse function of f.

10. Let $f : \mathbb{C}\backslash\{0\} \longrightarrow \mathbb{C}\backslash\{0\}$ be the function defined by $f(z) = \dfrac{z}{\bar{z}}$ for any $z \in \mathbb{C}\backslash\{0\}$.

    (a) Is f injective? Why?

    (b) Is f surjective? Why?

11. Let $a, b, c, d \in \mathbb{C}$. Suppose $c \neq 0$ and $ad - bc \neq 0$.

    (a) Prove that for any $z \in \mathbb{C}$, $\dfrac{az + b}{cz + d} \neq \dfrac{a}{c}$.

    (b) Define the function $f : \mathbb{C}\backslash\{-d/c\} \longrightarrow \mathbb{C}\backslash\{a/c\}$ by $f(z) = \dfrac{az + b}{cz + d}$ for any $z \in \mathbb{C}\backslash\{-d/c\}$.

       i. Verify that f is injective.

       ii. Verify that f is surjective.

       iii. Write down the 'formula of definition' of the inverse function of f.

12. We introduce the definition below:

    - *Let D be a subset of $\mathbb{C}$, and $f : D \longrightarrow \mathbb{C}$ be a function. Let $\zeta \in D$. ζ is said to be a* **zero of f in D** *if $f(\zeta) = 0$.*

    In this question, you are supposed to be familiar with the notion of continuity in the calculus of one real variable. You may take for granted the validity of **Bolzano's Intermediate Value Theorem***:*

    - *Let $a, b \in \mathbb{R}$, with $a < b$, and $f : [a, b] \longrightarrow \mathbb{R}$ be a function. Suppose f is continuous on $[a, b]$. Further suppose $f(a)f(b) < 0$. Then f has a zero in (a, b).*

    (a)♣ Let $p, q, r \in \mathbb{R}$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^3 + px^2 + qx + r$ for any $x \in \mathbb{R}$.
    You may take for granted that f is continuous on $\mathbb{R}$.
    Define $b = 1 + 2(|p| + |q| + |r|)$, and $a = -b$.

       i. Prove that $f(b) \geq \dfrac{b^3}{2}$ and $f(a) \leq -\dfrac{b^3}{2}$.

       ii. Hence apply Bolzano's Intermediate Value Theorem to deduce that f has a zero in (a, b).

    (b)♢ Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a cubic polynomial function with real coefficients. Prove that g is surjective.

13. Let A be the subset of $\mathbb{N}$ defined by $A = \{0, 1, 2, 3, 4, 5\}$, and $f : A \longrightarrow A$ be the function defined by $f(0) = 1$, $f(1) = 1$, $f(2) = 2$, $f(3) = 2$, $f(4) = 2$, $f(5) = 5$.

    Consider each of the sets below. Where it is not the empty set, list every element of the set concerned, each element exactly once. Where it is the empty set, write '*it is the empty set*'.

    (a) $f(\{1, 2, 3\})$

    (b) $f^{-1}(\{1\})$

    (c) $f^{-1}(\{3, 4\})$

    (d) $f(\{0, 2, 4\})$

    (e) $f^{-1}(\{0, 2, 4\})$

    (f) $f(f^{-1}(\{0, 2, 4\}))$

    (g) $f^{-1}(f(\{0, 2, 4\}))$

    (h) $(f \circ f)(\{0, 2, 4\})$

    (i) $(f \circ f)^{-1}(\{0, 2, 4\})$

14. Let $f : \mathbb{R}\backslash\{0, 2\} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{2}{x(x-2)} + 1$ for any $x \in \mathbb{R}\backslash\{0, 2\}$.

Write down the respective values of the numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \kappa, \lambda, \mu, \nu, \xi, \rho, \sigma, \tau, \varphi, \psi, \omega$, so that the set equalities below hold. You are not required to justify your answer.

(a) $f((0, 2)) = (-\infty, \alpha]$.

(b) $f([4, +\infty)) = (\beta, \gamma]$.

(c) $f((1, 3)\backslash\{2\}) = (-\infty, \delta) \cup (\varepsilon, +\infty)$.

(d) $f^{-1}(\{3\}) = \{\zeta, \eta\}$.

(e) $f^{-1}([3, 9]) = [\theta, \kappa] \cup [\lambda, \mu]$.

(f) $f^{-1}([-3, 0]) = [\nu, \xi]$.

(g) $f^{-1}([-3, 1]) = [\rho, \sigma]$.

(h) $f^{-1}([-3, 3]) = (-\infty, \tau] \cup [\varphi, \psi] \cup [\omega, +\infty)$.

15. Define the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x^{-2} & \text{if} \quad x < -1 \\ -1 & \text{if} \quad x = -1 \\ -x & \text{if} \quad -1 < x < 0 \\ 2 & \text{if} \quad x = 0 \\ 2x^2 + 1 & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad x = 1 \\ 1 + x^{-1} & \text{if} \quad x > 1 \end{cases} .$$

Write down the respective values of the numbers α, β, γ, δ, ε, ζ, η, θ, κ, so that the set equalities below hold. You are not required to justify your answer. (But it may help if you draw the graph of f first.)

(a) $f(\mathbb{R}) = ([\alpha, \beta)\backslash\{\gamma\}) \cup \{\delta\}$.

(b) $f^{-1}([0.25, 3]) = ([\varepsilon, \zeta]\backslash\{\eta\}) \cup ([\theta, +\infty)\backslash\{\kappa\})$.

16. *You are not required to justify your answers in this question.*

Let $a, b \in \mathbb{R}$, and $f : [0, 5] \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} -\dfrac{12}{(x+1)(x-3)} & \text{if} \quad 0 \leq x < 3 \\ a & \text{if} \quad x = 3 \\ -(x-3)(x-5) & \text{if} \quad 3 < x \leq 5 \text{ and } x \neq 4 \\ b & \text{if} \quad x = 4 \end{cases} .$$

Suppose $f(3) < f(4)$. Further suppose that $f^{-1}(\{2\}) \neq \emptyset$ and $f^{-1}(\{3\})$ has exactly two elements.

(a) What are the respective values of a, b?

(b) Name all two elements of $f^{-1}(\{3\})$.

(c) What are the numbers $\alpha, \beta, \gamma, \delta$ for which the set equality $f([2, 4)) = (\alpha, \beta) \cup \{\gamma\} \cup [\delta, +\infty)$ holds?

17. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = 2x^4 - 4$ for any $x \in \mathbb{R}$.

(a) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the equality $f([1, 2]) = [-2, 28]$. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

Write $S = [1, 2]$.

- [We want to verify the statement (†): '*for any* y*, if* $y \in f(S)$ *then* $y \in [-2, 28]$*.*']

  Pick any y. ________(I)________ .

  Then by the definition of $f(S)$, ____________(II)____________ .

  For the same x, since $x \in S$, we have $1 \leq x \leq 2$.

  Since $x \geq 1$, we have __________(III)__________ .

  ________(IV)________ we have $y = f(x) = 2x^4 - 1 \leq 2 \cdot 2^4 - 4 = 28$.

  Therefore $-2 \leq y \leq 28$.

  Hence $y \in [-2, 28]$.

- [We want to verify the statement (‡): '*for any* y*, if* $y \in [-2, 28]$ *then* $y \in f(S)$*.*']

  Pick any y. Suppose $y \in [-2, 28]$. Then $-2 \leq y \leq 28$.

  [We want to verify that for this y, there exists some $x \in S$ such that $y = f(x)$.]

  ________(V)________ .

  We verify that $x \in S$:

  ∗ Since $y \geq -2$, we have $\dfrac{y+4}{2} \geq 1$. Then ________(VI)________ .

  __________(VII)__________ . Then __________(VIII)__________ .

  Therefore $1 \leq x \leq 2$. Hence $x \in [1, 2] = S$.

  For the same x, we have __________(IX)__________ .

  Then, for the same x, y, we have $x \in S$ and $y = f(x)$.

  Hence by the definition of $f(S)$, ________(X)________ .

It follows that $f(S) = [-2, 28]$.

(b) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the equality $f^{-1}([-6, 4]) = [-\sqrt{2}, \sqrt{2}]$. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

Write $U = [-6, 4]$.

- [We want to verify the statement (†): '*for any x, if $x \in f^{-1}(U)$ then $x \in [-\sqrt{2}, \sqrt{2}]$.*']

  Pick any x. ______(I)______ .

  Then by the definition of $f^{-1}(U)$, ______(II)______

  For the same y, since ______(III)______ , we have $-6 \leq y \leq 4$.

  Since $y \geq -6$, we have $2x^4 - 4 = f(x) = y \geq -6$. Then $x^4 \geq -1$. (This provides no information other than re-iterating '$x \in \mathbb{R}$'.)

  Since $y \leq 4$, we have ______(IV)______ . Then $x^4 \leq 4$. Since $x \in \mathbb{R}$, we have $-\sqrt{2} \leq x \leq \sqrt{2}$.

  Then $x \in [-\sqrt{2}, \sqrt{2}]$.

- [We want to verify the statement (‡): '*for any x, if $x \in [-\sqrt{2}, \sqrt{2}]$ then $x \in f^{-1}(U)$.*']

  Pick any x. ______(V)______ . Then $-\sqrt{2} \leq x \leq \sqrt{2}$.

  [We want to verify that for this x, there exists some $y \in U$ such that $y = f(x)$.]

  ______(VI)______ . We verify that $y \in U$:

  * Since $-\sqrt{2} \leq x \leq \sqrt{2}$, we have $x^4 \leq 4$. Then ______(VII)______ .

    Since $x \in \mathbb{R}$, we have $x^4 \geq 0 \geq -2$. Then ______(VIII)______ .

    Therefore $-6 \leq y \leq 4$. Hence $y \in [-6, 4] = U$.

  Then, for the same x, y, we have $y = f(x)$ ______(IX)______ $y \in U$.

  Hence by the definition of $f^{-1}(U)$, ______(X)______.

It follows that $f^{-1}(U) = [-\sqrt{2}, \sqrt{2}]$.

18. *You are not required to justify your answers in this question. In each part, you are only required to give one correct answer, although there are different correct answers.*

   (a) Let $A = (-1, 1]$, $B = [-2, 2)$, $G = \{(x, x) \mid x \leq 0\}$, $H = \{(x, x+1) \mid x \geq 0\}$ and $F = (A \times B) \cap (G \cup H)$.
   Name some appropriate $(p, q), (s, t) \in A \times B$, if such exist, for which the ordered triple $(A, B, (F \backslash \{(p, q)\}) \cup \{(s, t)\})$ is a function from A to B.

   (b) Let $A = [0, 2]$, $G = \{(x, x^2) \mid 0 \leq x \leq 1\}$, $H = \{(x, 3 - x) \mid 1 \leq x < 2\}$ and $F = A^2 \cap (G \cup H)$.
   Name some appropriate $(p, q), (s, t) \in A^2$, if such exist, for which the ordered triple $(A, A, (F \backslash \{(p, q)\}) \cup \{(s, t)\})$ is an injective function from A to A.

   (c) Let $A = [0, +\infty)$ and E, F be the subsets of $\mathbb{R}^2$ defined respectively by $E = \{(x, x^{-1}) \mid 0 < x \leq 1\}$, $F = \{(x, 2x^{-2}) \mid x \geq 1\}$.
   Name some appropriate $(m, n), (p, q) \in A^2$, if such exist, for which the ordered triple $(A, A, (E \cup F \cup \{(m, n)\}) \backslash \{(p, q)\})$ is a surjective function from A to A.