MATH1050 Exercise 8 Supplement (Answers)

1. (a) ——
   (b) 0 is a lower bound of $\{b_n\}_{n=0}^{\infty}$
2. ——
3. ——
4. (a) 2 is the greatest element of S.
   (b) ——
   (c) S is bounded below by 1 in $\mathbb{R}$.
5. (a) 2 is the greatest element of S.
   (b) S has no least element.
   (c) S is bounded below by 0 in $\mathbb{R}$.
6. ——
7. ——
8. ——
9. ——
10. (a) *Hint.* When f is not 'identically zero' on $[a, b]$, study the quadratic function $F : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $F(t) = \int_a^b (tf(u) + g(u))^2 du$ for any $t \in \mathbb{R}$.
    (b) ——
11. ——
12. (a) ——
    (b) ——
    (c) *Hint.* Write $A = \int_0^{\frac{1}{2}} (f'(t))^2 dt$, $B = \int_{\frac{1}{2}}^1 (f'(t))^2 dt$. By the result in part (b.i), we have $(f(x))^2 \leq Ax$ for any $x \in [0, \frac{1}{2}]$. By the result in part (b.ii), we have $(f(x))^2 \leq B(1 - x)$ for any $x \in [\frac{1}{2}, 1]$. So what happens?
13. ——
14. ——
15. ——
16. ——
17. ——
18. ——
19. ——
20. (a) ——
    (b) i. $R = 1$, $S = 1$, $T = 1$.
        ii. ——
        iii. ——
        iv. ——
    (c) ——

    Remark. *Beyond* MATH1050 *and towards mathematical analysis.* According to the Bounded-Monotone Theorem and the Sandwich Theorem, for each $\alpha > 0$, the infinite sequences $\{a_n(\alpha)\}_{n=2}^{\infty}$, $\{b_n(\alpha)\}_{n=2}^{\infty}$, $\{c_n(\alpha)\}_{n=2}^{\infty}$ defined by

    $$a_n = \left(1 + \frac{\alpha}{n}\right)^n, \qquad b_n = \sum_{k=0}^{n} \frac{\alpha^k}{k!}, \qquad c_n = \left(1 - \frac{\alpha^2}{2n}\right) \sum_{k=0}^{n} \frac{\alpha^k}{k!}, \quad \text{whenever } n \geq 2$$

    'converges' to the same limit. This limit is the number e^{α}.
21. ——
22. ——
23. ——