1. Let $\{a_n\}_{n=0}^{\infty}$ be the infinite sequence of positive real numbers recursively defined by

$$\begin{cases} a_0 = \sqrt{2} \\ a_{n+1} = \sqrt{2+a_n} \quad \text{for each } n \in \mathbb{N} \end{cases}$$

   (a) i. Prove that $\{a_n\}_{n=0}^{\infty}$ is strictly increasing.

   ii. Prove that $a_n < 2$ for each $n \in \mathbb{N}$.

   iii. Prove that $\dfrac{2-a_1}{2-a_0} \leq \dfrac{2-\sqrt{2}}{2}$.

   iv. Prove that $2 - a_n \leq \left(\dfrac{2-\sqrt{2}}{2}\right)^n (2-\sqrt{2})$ for each $n \in \mathbb{N}$.

   (b) Now let $\{b_n\}_{n=0}^{\infty}$ be the infinite sequence of positive real numbers $b_n = \dfrac{a_0 a_1 a_2 \cdot \ldots \cdot a_n}{2^{n+1}}$ for each $n \in \mathbb{N}$.

   Name a lower bound for $\{b_n\}_{n=0}^{\infty}$, and justify your answer. Also prove that $\{b_n\}_{n=0}^{\infty}$ is strictly decreasing.

   Remark. It follows from the Bounded-Monotone Theorem that $\lim_{n\to\infty} b_n$ exists. It turns out that $\lim_{n\to\infty} b_n = \dfrac{2}{\pi}$. This is known as **Vieta's formula for π**.

2. Let $0 < \alpha < 1$ and $\{a_n\}_{n=0}^{\infty}$ be the infinite sequence of positive real numbers recursively defined by

$$\begin{cases} a_0 = \alpha \\ a_{n+1} = \sqrt{\dfrac{1+a_n}{2}} \quad \text{for each } n \in \mathbb{N} \end{cases}$$

   (a) i. Prove that $0 < a_n < 1$ for each $n \in \mathbb{N}$.

   ii. Prove that $\{a_n\}_{n=0}^{\infty}$ is strictly increasing.

   (b) Let $\{b_n\}_{n=0}^{\infty}$ be the infinite sequence of positive real numbers defined by $b_n = \sqrt{\dfrac{1-a_n}{2}}$ for each $n \in \mathbb{N}$.

   i. Prove that $\dfrac{b_n}{2} < b_{n+1} < \dfrac{b_n}{\sqrt{2}}$ for each $n \in \mathbb{N}$.

   ii. Hence, or otherwise, deduce that $0 < b_n < \dfrac{\sqrt{1-\alpha}}{(\sqrt{2})^{n+1}}$ and $1 - \dfrac{1-\alpha}{2^n} < a_n < 1$ for each $n \in \mathbb{N}$.

3. For any $n \in \mathbb{N}\backslash\{0\}$, define $b_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k}$.

   (a) Verify that $\{b_n\}_{n=1}^{\infty}$ is strictly increasing.

   (b) i. Prove that for any $m \in \mathbb{N}$, $b_{2^{m+1}} - b_{2^m} \geq \dfrac{1}{2}$.

   ii.$^\diamond$ Hence deduce that $\{b_n\}_{n=1}^{\infty}$ is not bounded above in $\mathbb{R}$. (*Hint.* Apply the Telescopic Method.)

   Remark. In the argument, you may apply the apparently 'obviously true' statement below:

   - *The set $\mathbb{N}$ is not bounded above in $\mathbb{R}$.*

4. Let $S = \left\{ x \,\middle|\, x = \dfrac{n+2}{n+1} \text{ for some } n \in \mathbb{N} \right\}$.

   (a) Prove that S has a greatest element.

   (b) Prove that S does not have any least element.

   (c) Prove that S is bounded below in $\mathbb{R}$.

5. Let $S = \left\{ x \in \mathbb{R} : x = \dfrac{1}{3^m} + \dfrac{1}{5^n} \text{ for some } m, n \in \mathbb{N} \right\}$.

   (a) Does S have any greatest element? Why?

   (b) Does S have any least element? Why?

(c) Is S bounded below in $\mathbb{R}$? Why?

6. *The various parts in this question are concerned with applications of the Cauchy-Schwarz Inequality. They are independent of each other.*

(a) Let $a_1, a_2, \cdots, a_n$ be positive real numbers. Prove the statements below:

i. ${a_1}^2 + {a_2}^2 + \cdots + {a_n}^2 \geq a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1$.

ii. ${a_1}^2 + {a_2}^2 + \cdots + {a_n}^2 = a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1$ *iff* $a_1 = a_2 = \cdots = a_n$.

(b) Let $a_1, a_2, \cdots, a_n$ be non-zero real numbers. Prove the statements below:

i. $n^2 \leq \left(\sum_{k=1}^{n} {a_k}^2\right)\left(\sum_{k=1}^{n} \frac{1}{{a_k}^2}\right)$.

ii. $n^2 = \left(\sum_{k=1}^{n} {a_k}^2\right)\left(\sum_{k=1}^{n} \frac{1}{{a_k}^2}\right)$ *iff* $|a_1| = |a_2| = \cdots = |a_n|$.

Remark. How about an argument using the Arithmetico-geometrical Inequality?

(c) Prove the statements below:

i. *Suppose* $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n, c_1, c_2, \cdots, c_n, d_1, d_2, \cdots, d_n$ *are real numbers. Then*

$$\left(\sum_{j=1}^{n} a_jb_jc_jd_j\right)^4 \leq \left(\sum_{j=1}^{n} {a_j}^4\right)\left(\sum_{j=1}^{n} {b_j}^4\right)\left(\sum_{j=1}^{n} {c_j}^4\right)\left(\sum_{j=1}^{n} {d_j}^4\right).$$

ii. *Suppose* $r_1, r_2, \cdots, r_n, s_1, s_2, \cdots, s_n, t_1, t_2, \cdots, t_n$ *are non-negative real numbers. Then*

$$\left(\sum_{j=1}^{n} r_js_jt_j\right)^3 \leq \left(\sum_{j=1}^{n} {r_j}^3\right)\left(\sum_{j=1}^{n} {s_j}^3\right)\left(\sum_{j=1}^{n} {t_j}^3\right).$$

7. (a) Let $a_1, a_2, \cdots, a_n$ be positive real numbers. Let m be a non-negative integer.

Prove that $\left(\sum_{j=1}^{n} {a_j}^{m+1}\right)^2 \leq \left(\sum_{j=1}^{n} {a_j}^{m}\right)\left(\sum_{j=1}^{n} {a_j}^{m+2}\right)$.

Remark. You may need the Cauchy-Schwarz Inequality.

(b)$^\diamond$ Let $b_1, b_2, \cdots, b_n$ be positive real numbers. Suppose $\sum_{j=1}^{n} b_j = 1$.

By applying the results in the previous part, or otherwise, prove that $\sum_{j=1}^{n} {b_j}^p \leq n\sum_{j=1}^{n} {b_j}^{p+1}$ for each non-negative integer p.

(c)$^\diamond$ Let $c_1, c_2, \cdots, c_n$ be positive real numbers.

By applying the results in the previous part, or otherwise, prove that $\left(\sum_{j=1}^{n} c_j\right)\left(\sum_{j=1}^{n} {c_j}^r\right) \leq n\sum_{j=1}^{n} {c_j}^{r+1}$ for each non-negative integer r.

8. *In this question, you may need apply the Cauchy-Schwarz Inequality more than once.*

(a) Let $a_1, a_2, \cdots, a_n$ be real numbers. Prove that $\frac{1}{n}\left(\sum_{k=1}^{n} a_k\right)^2 \leq \sum_{k=1}^{n} {a_k}^2$.

(b) Let $b_1, b_2, \cdots, b_n, c_1, c_2, \cdots, c_n$ are real numbers. Further suppose that $b_1, b_2, \cdots, b_n$ are positive.

Prove that $\left(\sum_{k=1}^{n} b_kc_k\right)^2 \leq \left(\sum_{k=1}^{n} b_k\right)\left(\sum_{j=1}^{n} b_j{c_j}^2\right)$.

(c)$^\diamond$ Let $r \geq 2$, and $x_1, x_2, \cdots, x_n$ be real numbers which are not all zero.

By applying the results in the previous part, or otherwise, prove that $\left(\sum_{k=1}^{n} \frac{x_k}{r^k}\right)^2 < \sum_{k=1}^{n} \frac{{x_k}^2}{r^k}$.

9. (a) Let $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ be non-zero real numbers.

i. Let p, q be real numbers. Suppose $p \leq \dfrac{b_k}{a_k} \leq q$ for each $k = 1, 2, \cdots, n$.

Prove that $(p+q)\displaystyle\sum_{k=1}^{n} a_k b_k \geq \sum_{k=1}^{n} {b_k}^2 + pq\sum_{k=1}^{n} {a_k}^2$.

ii. Let m, M be real numbers. Suppose $0 < m \leq a_k \leq M$ and $0 < m \leq b_k \leq M$ for each $k = 1, 2, \cdots, n$. By applying the result in the previous part, or otherwise, prove that

$$\left(\sum_{k=1}^{n} {a_k}^2\right)\left(\sum_{k=1}^{n} {b_k}^2\right) \leq \frac{1}{4}\left(\frac{M}{m} + \frac{m}{M}\right)^2 \left(\sum_{k=1}^{n} a_k b_k\right)^2.$$

(b)$^{\diamond}$ Apply the results in the previous part, together with the Cauchy-Schwarz Inequality, or otherwise, prove that for each positive integer n,

$$\left(n + \frac{1}{9}\right)^2 < \left[\sum_{k=1}^{n}\left(1 + \frac{1}{3^k}\right)^2\right]\left[\sum_{k=1}^{n}\left(1 - \frac{1}{3^{k+1}}\right)^2\right] < \frac{169}{144}\left(n + \frac{1}{3}\right)^2.$$

10.$^{\diamond}$ *In this question you are assumed to be familiar with calculus of one real variable.*

Take for granted the validity of the result below about definite integrals:

- *Let a, b be real numbers, with $a < b$, and let h be a real-valued function of one real variable whose domain contains the interval $[a, b]$. Suppose h is continuous on $[a, b]$. Further suppose that $h(x) \geq 0$ for any $x \in [a, b]$. Then $\displaystyle\int_a^b h(t)dt \geq 0$. Moreover, equality holds iff $h(x) = 0$ for any $x \in [a, b]$.*

(a) Prove the statement below, which is the 'inequality part' in the result known as **Cauchy-Schwarz Inequality for definite integrals**:

- *Suppose $f, g : (\alpha, \beta) \longrightarrow \mathbb{R}$ are continuous functions. Then, whenever $\alpha < a < b < \beta$,*

$$\left|\int_a^b f(u)g(u)du\right| \leq \left(\int_a^b |f(u)|^2 du\right)^{\frac{1}{2}} \left(\int_a^b |g(u)|^2 du\right)^{\frac{1}{2}}.$$

(b) Prove the statement below, which is the 'inequality part' in the result known as **Triangle Inequality for definite integrals**:

- *Suppose $f, g : (\alpha, \beta) \longrightarrow \mathbb{R}$ are continuous functions. Then, whenever $\alpha < a < b < \beta$,*

$$\sqrt{\int_a^b |f(u) + g(u)|^2 du} \leq \sqrt{\int_a^b |f(u)|^2 du} + \sqrt{\int_a^b |g(u)|^2 du}.$$

11. *In this question you are assumed to be familiar with calculus of one real variable.*

Let $a, b \in \mathbb{R}$, with $a < b$, and f be a real-valued function of one real variable which is twice-continuously differentiable on an open interval which contains the closed and bounded interval $[a, b]$ entirely. Suppose $f(a) = f(b) = 0$.

(a) Verify that $\displaystyle\int_a^b f(x)f''(x)dx = -\int_a^b (f'(x))^2 dx$.

(b) Here we suppose that $\displaystyle\int_a^b (f(x))^2 dx = 1$.

i. Prove that $\displaystyle\int_a^b xf(x)f'(x)dx = -\frac{1}{2}$.

ii.$^{\diamond}$ By applying the Cauchy-Schwarz Inequality, or otherwise, deduce that

$$\left(\int_a^b (f'(x))^2 dx\right)\left(\int_a^b u^2 (f(u))^2 du\right) \geq \frac{1}{4}.$$

(c) Here we no longer suppose that $\displaystyle\int_a^b (f(x))^2 dx = 1$. We only suppose that f is not constant on $[a, b]$.

Take for granted that $\displaystyle\int_a^b |f(x)|^2 dx > 0$.

i.♡ Prove that $\left(\int_a^b (f'(x))^2 dx\right)\left(\int_a^b u^2 (f(u))^2 du\right) \geq \frac{1}{4}\left(\int_a^b (f(x))^2 dx\right)^2$.

ii.♣ Hence, or otherwise, prove that $\left(\int_a^b (f''(x))^2 dx\right)\left(\int_a^b u^2 (f(u))^2 du\right)^2 \geq \frac{1}{16}\left(\int_a^b (f(x))^2 dx\right)^3$.

(*Hint.* At some stage of the argument, you may need the Cauchy-Schwarz Inequality.)

12. *In this question you are assumed to be familiar with calculus of one real variable.*

Take for granted the validity of the result below about definite integrals:

- *Let a, b be real numbers, with $a < b$, and let g, h be real-valued functions of one real variable whose domains contain the interval $[a, b]$. Suppose g, h are continuous on $[a, b]$. Further suppose that $g(x) \leq h(x)$ for any $x \in [a, b]$. Then $\int_a^b g(t)dt \leq \int_a^b h(t)dt$.*

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function. Suppose f is continuously differentiable function on $\mathbb{R}$. Suppose $f(0) = 0$ and $f(1) = 0$.

(a) Prove that $f(x) = \int_0^x f'(t)dt = -\int_x^1 f'(t)dt$ for any $x \in [0, 1]$.

(b)♣ By applying the Cauchy-Schwarz Inequality, or otherwise, prove the statements below:

i. $(f(x))^2 \leq x\int_0^{\frac{1}{2}} (f'(t))^2 dt$ for any $x \in [0, \frac{1}{2}]$.

ii. $(f(x))^2 \leq (1-x)\int_{\frac{1}{2}}^1 (f'(t))^2 dt$ for any $x \in [\frac{1}{2}, 1]$.

(c)♣ Hence, or otherwise, prove that $\int_0^1 (f(x))^2 dx \leq \frac{1}{8}\int_0^1 (f'(x))^2 dx$.

13. *The various parts in this question are concerned with applications of the Arithmetico-geometrical Inequality. They are independent of each other.*

(a) Let n be an integer greater than 1. Suppose $b_1, b_2, \cdots, b_n$ are positive real numbers.

Prove that $\frac{b_1}{b_2} + \frac{b_2}{b_3} + \cdots + \frac{b_{n-1}}{b_n} + \frac{b_n}{b_1} \geq n$.

(b) i. Let n be a positive integer. Prove that $\frac{n+2}{n+1} > \left(\frac{n+1}{n}\right)^{n/(n+1)}$.

ii. Hence deduce that $\left(1 + \frac{1}{m+k}\right)^{m+k} > \left(1 + \frac{1}{m}\right)^m$ whenever m, k are positive integers.

(c) Let n be a positive integer.

i. Prove that $\frac{1}{n}\left[\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}\right] > \frac{1}{\sqrt[n]{(n!)^2(n+1)}}$.

ii. Hence deduce that $(n!)^2 > (n+1)^{n-1}$.

(d) Let $a_1, a_2, \cdots, a_n$ be n positive numbers, and $S = \sum_{j=1}^n a_j$.

i. Prove that $\sum_{k=1}^n \frac{S}{S - a_k} \geq \frac{n^2}{n-1}$.

ii. Hence, or otherwise, deduce that $\sum_{k=1}^n \frac{a_k}{S - a_k} \geq \frac{n}{n-1}$.

(e) Let the angles at the vertices A, B, C in $\triangle ABC$ be α, β, γ respectively. Suppose each angle in $\triangle ABC$ is an acute angle.

i. Prove that $\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1 - 2\cos(\alpha)\cos(\beta)\cos(\gamma)$.

Remark. At some stage you may need express $\cos^2(\mu) + \cos^2(\nu)$ in terms of $\cos(2\mu), \cos(2\nu)$.

ii. Prove that $\cos^2(\alpha)\cos^2(\beta)\cos^2(\gamma) < \frac{1}{27}$.

14. *Here we are going to re-prove the Arithmetico-geometrical Inequality.*

(a) Let a, b be real numbers. Suppose $a \leq 1 \leq b$.
Prove that $a + b \geq ab + 1$. Also prove that equality holds iff ($a = 1$ or $b = 1$).

(b)♣ Apply mathematical induction to verify the statement below:

(†) *Let n be an integer greater than 1. Suppose $c_1, c_2, \cdots, c_n$ are positive real numbers. Further suppose $c_1 c_2 \cdot \ldots \cdot c_n = 1$. Then $c_1 + c_2 + \cdots + c_n \geq n$. Equality holds iff $c_1 = c_2 = \cdots = c_n$.*

Remark. At some stage of the 'inductive argument', you may need the result in part (a).

(c)♢ By applying the result in the previous part, or otherwise, show that the statement below is true:

(‡) *Let n be an integer greater than 1. Suppose $a_1, a_2, \cdots, a_n$ are positive real numbers. Then $(a_1 a_2 \cdot \ldots \cdot a_n)^{\frac{1}{n}} \leq \dfrac{a_1 + a_2 + \cdots + a_n}{n}$. Equality holds iff $a_1 = a_2 = \cdots = a_n$.*

15. *Here we are going to re-prove the Arithmetico-geometrical Inequality.*

(a) Let n be an integer greater than 1. Let a, b be non-negative real numbers. Prove that $(a+b)^n \geq a^n + na^{n-1}b$. Also prove that equality holds iff $b = 0$.

(b) Let $\{x_n\}_{n=1}^{\infty}$ be an increasing infinite sequence of positive real numbers.

For each positive integer n, define $A_n = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$, $G_n = (x_1 x_2 \cdot \ldots \cdot x_n)^{\frac{1}{n}}$.

i. Let m be a positive integer.

A.♢ Prove that $A_{m+1} \geq A_m$.

B. Express x_{m+1} in the form $P_m A_{m+1} - Q_m A_m$ for some appropriate positive integers P_m, Q_m (whose values depend on m).

C.♣ Prove that ${A_{m+1}}^{m+1} \geq {A_m}^m x_{m+1}$. (Hint: Start by re-expressing A_{m+1} as $A_m + (A_{m+1} - A_m)$, and apply the result in part (a).)

ii. Hence, or otherwise, prove that the statement below is true:

- *Let n be a positive integer. $A_n \geq G_n$. Equality holds iff $x_1 = x_2 = \cdots = x_n$.*

16. (a) Verify the statements below:

i. *Let $a, b \in \mathbb{R}$. Suppose $0 \leq a < b$. Then $\dfrac{a}{1+a} \leq \dfrac{b}{1+b}$.*

ii. *Let $a, b \in \mathbb{R}$. Suppose a, b are non-negative. Then $\dfrac{a+b}{1+a+b} \leq \dfrac{a}{1+a} + \dfrac{b}{1+b}$.*

(b) Applying mathematical induction, or otherwise, prove the statement below:

- *Let $n \in \mathbb{N}\backslash\{0\}$. Suppose $x_1, x_2, \cdots, x_n$ are non-negative real numbers. Then*

$$\frac{x_1 + x_2 + \cdots + x_n}{1 + x_1 + x_2 + \cdots + x_n} \leq \sum_{j=1}^{n} \frac{x_j}{1 + x_j}.$$

(c) Hence, or otherwise, prove the statement below:

- *Let $n \in \mathbb{N}\backslash\{0\}$. Suppose $c_1, c_2, \cdots, c_n$ are non-negative real numbers. Then*

$$\frac{c_1 c_2 \cdots c_n}{1 + c_1 c_2 \cdots c_n} \leq \frac{{c_1}^n + {c_2}^n + \cdots + {c_n}^n}{n + {c_1}^n + {c_2}^n + \cdots + {c_n}^n} \leq \sum_{j=1}^{n} \frac{{c_j}^n}{n + {c_j}^n}.$$

17. (a) Let n, k be positive integers. Suppose $k \leq n$. Verify that $\dfrac{1}{n^k}\begin{pmatrix} n \\ k \end{pmatrix} \geq \dfrac{1}{k!}$.

(b) Applying the Arithmetico-Geometrical Inequality together with the Binomial Theorem, or otherwise, prove the statement below:

- *Let $a_1, a_2, \cdots, a_n$ be positive real numbers. Suppose $s = \sum_{j=1}^{n} a_n$. Then $\prod_{k=1}^{n}(1 + a_k) \leq \sum_{r=0}^{n} \dfrac{s^r}{r!}$.*

(c) Let $x \in \mathbb{R}$. Suppose $0 < x < 1$. For any $n \in \mathbb{N}\backslash\{0\}$, define $b_n = \prod_{k=1}^{n}(1 + x^k)$.

i. Prove that $\{b_n\}_{n=1}^{\infty}$ is strictly increasing.

ii. *Take for granted that* $\displaystyle\sum_{r=0}^{n} \frac{p^r}{r!} \leq e^p$ *for any* $p > 0$ *for any* $n \in \mathbb{N}$.

Prove that $\{b_n\}_{n=1}^{\infty}$ is bounded above by $e^{x/(1-x)}$.

18. Let p be a positive integer.

(a) Let x be a real number. Suppose $0 \leq x \leq 1$.

i. Prove that $1 + x^p \geq x^k + x^{p-k}$ for each $k = 0, 1, 2, \cdots, p$.

ii.$^\diamond$ Hence, or otherwise, deduce that $(1 + x)^p \leq 2^{p-1}(1 + x^p)$.

(b) Let u, v be positive real numbers. Prove that $\left(\dfrac{u+v}{2}\right)^p \leq \dfrac{u^p + v^p}{2}$.

(c)$^\diamond$ Apply mathematical induction to justify the statement ($\uparrow$) below:

($\uparrow$) *Let n be a non-negative integer. Suppose $a_1, a_2, \cdots, a_{2^n}$ are positive real numbers. Then*

$$\left(\frac{a_1 + a_2 + \cdots + a_{2^n}}{2^n}\right)^p \leq \frac{a_1{}^p + a_2{}^p + \cdots + a_{2^n}{}^p}{2^n}.$$

(d)$^\clubsuit$ Hence, or otherwise, prove that the statement ($\Uparrow$) below is true:

($\Uparrow$) *Let n be a positive integer. Suppose $a_1, a_2, \cdots, a_n$ be positive real numbers. Then*

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^p \leq \frac{a_1{}^p + a_2{}^p + \cdots + a_n{}^p}{n}.$$

Remark. Suppose p is a positive integer and $a_1, a_2, \cdots, a_n$ are n positive real numbers. Then the number $\sqrt[p]{\dfrac{a_1{}^p + a_2{}^p + \cdots + a_n{}^p}{n}}$ is called the **mean power of** $a_1, a_2, \cdots, a_n$ **of order** p. (When $p = 1$, this number is the arithmetic mean of $a_1, a_2, \cdots, a_n$; when $p = 2$, this number is more often called the root-square-mean of $a_1, a_2, \cdots, a_n$.) What has proved here is that the mean power of a collection of finitely many positive real numbers of order p is greater than or equal to the arithmetic mean of the same collection of numbers.

19. (a) Let x, y be positive real numbers. Prove that $\sqrt{(1 + x)(1 + y)} \geq 1 + \sqrt{xy}$.

(b)$^\diamond$ Apply mathematical induction to justify the statement ($\uparrow$) below:

($\uparrow$) *Let $n \in \mathbb{N}$. Suppose $a_1, a_2, \cdots, a_{2^n}$ are positive real numbers. Then*

$$\sqrt[2^n]{(1 + a_1)(1 + a_2) \cdot \ldots \cdot (1 + a_{2^n})} \geq 1 + \sqrt[2^n]{a_1 a_2 \cdot \ldots \cdot a_{2^n}}$$

.

(c)$^\clubsuit$ Hence, or otherwise, prove that the statement ($\Uparrow$) below is true:

($\Uparrow$) *Let $n \in \mathbb{N} \backslash \{0, 1\}$. Suppose $a_1, a_2, \cdots, a_n$ are positive real numbers. Then*

$$\sqrt[n]{(1 + a_1)(1 + a_2) \cdot \ldots \cdot (1 + a_n)} \geq 1 + \sqrt[n]{a_1 a_2 \cdot \ldots \cdot a_n}$$

.

(d) Hence, or otherwise, deduce the statement ($\sharp$):

($\sharp$) *Let $n \in \mathbb{N} \backslash \{0, 1\}$. Suppose $u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n$ are positive real numbers. Then*

$$\sqrt[n]{(u_1 + v_1)(u_2 + v_2) \cdot \ldots \cdot (u_n + v_n)} \geq \sqrt[n]{u_1 u_2 \cdot \ldots \cdot u_n} + \sqrt[n]{v_1 v_2 \cdot \ldots \cdot v_n}$$

20.$^\diamond$ Let α be a positive real number. For each $n \geq 2$, define

$$a_n = \left(1 + \frac{\alpha}{n}\right)^n, \qquad b_n = \sum_{k=0}^{n} \frac{\alpha^k}{k!}, \qquad c_n = \left(1 - \frac{\alpha^2}{2n}\right) \sum_{k=0}^{n} \frac{\alpha^k}{k!}.$$

(a) Show that $\{b_n\}_{n=2}^{\infty}$ is strictly increasing.

(b) i. By applying the Binomial Theorem, or otherwise, show that

$$a_n = R + S\alpha + \sum_{k=2}^{n} \frac{T\alpha^k}{k!} \cdot T \cdot \left(T - \frac{1}{n}\right)\left(T - \frac{2}{n}\right) \cdot \ldots \cdot \left(T - \frac{k-1}{n}\right)$$

whenever $n \geq 2$. Here R, S, T are positive integers which are independent of n. You have to determine the respective values of R, S, T explicitly.

ii. Deduce that $a_n < b_n$ whenever $n \geq 2$.

iii. Show that $b_n \leq b_{N-1} + \dfrac{\alpha^N}{(1-\alpha/N)\cdot(N!)}$ whenever $n \geq 2$ and $N > \alpha$.

iv. Also show that $c_n < a_n$ whenever $n \geq \dfrac{\alpha^2}{2}$.

(c) Show that $\{a_n\}_{n=2}^{\infty}$ is strictly increasing.

21.♡ Let S be a subset of $\mathbb{R}$ and $\sigma \in \mathbb{R}$. Suppose σ is an upper bound of S in $\mathbb{R}$. Prove that the two statements below are logically equivalent:

(†) For any $\beta \in \mathbb{R}$, if β is an upper bound of S in $\mathbb{R}$ then $\sigma \leq \beta$.

(‡) For any positive real number ε, there exists some $x \in S$ such that $\sigma - \varepsilon < x$.

Remark. We introduce/recall the definition for the notion of **least upper bound of a set** here:

Let S be a subset of $\mathbb{R}$ and $\sigma \in \mathbb{R}$. σ is said to be a least upper bound of S in $\mathbb{R}$ if both of the statements **(LU1)**, **(LU2)** *are true:*

(LU1) *σ is an upper bound of S in $\mathbb{R}$.*

(LU2) *For any $\beta \in \mathbb{R}$, if β is an upper bound of S in $\mathbb{R}$ then $\sigma \leq \beta$.*

We also introduce/recall the **Least-upper-bound Axiom** for the real number system:

Let S be a subset of $\mathbb{R}$. Suppose S is non-empty and is bounded above in $\mathbb{R}$. Then S has a least upper bound.

What we have established in this question is an equivalent formulation for the definition for 'least upper bound of a set': we may replace **(LU2)** by:

(LU2') *For any positive real number ε, there exists some $x \in S$ such that $\sigma - \varepsilon < x$.*

22. (a) Let $\alpha \in \mathbb{R}$, and $A = \{x \mid x < \alpha\}$. Verify the statements below:

i. *For any $x \in A$, $y \in \mathbb{R}$, if $y < x$ then $y \in A$.*

ii. *$A \neq \emptyset$.*

iii. *$A \neq \mathbb{R}$.*

iv. *For any $x \in A$, there exists $x' \in A$ such that $x < x'$.*

(b)♡ Let A be a subset of $\mathbb{R}$. Suppose Conditions (C1), (C2), (C3), (C4) are all satisfied:

(C1) *For any $x \in A$, $y \in \mathbb{R}$, if $y < x$ then $y \in A$.*

(C2) *$A \neq \emptyset$.*

(C3) *$A \neq \mathbb{R}$.*

(C4) *For any $x \in A$, there exists $x' \in A$ such that $x < x'$.*

Prove that there exists some $\alpha \in \mathbb{R}$ such that $A = \{x \mid x < \alpha\}$. (*You will need the Least-upper-bound Axiom at some stage in your argument.*)

Remark. What we have just proved is a **characterization of half-open intervals.**

23. (a) Prove the statements below:

i. *Let $\alpha, \beta \in \mathbb{Q}$. Suppose $\alpha < \beta$. Then there exists some $r \in \mathbb{Q}$ such that $\alpha < r < \beta$.*

ii. *Let $\alpha, \beta \in \mathbb{Q}$. Suppose $\alpha < \beta$. Then there exists some $u \in \mathbb{R}\backslash\mathbb{Q}$ such that $\alpha < r < \beta$.*

Remark. Don't think too hard.

(b)♡ Apply the Archimedean Principle and the Well-ordering Principle for Integers to prove the statement below:

(♮) *Let $\alpha, \beta \in \mathbb{R}$. Suppose $\beta > \alpha > 0$. Then there exists some $r \in \mathbb{Q}$ such that $\alpha < r < \beta$.*

(*Hint.* Is it guaranteed that there is some positive integer, say, N for which the interval (α, β) will contain at least two distinct fractions with denominator N and with integral numerators? If there is indeed such an integer N, is it true that $\frac{1}{N} < \beta - \alpha$?)

(c) Hence, or otherwise, prove the statements below:

i.♣ *Let $\alpha, \beta \in \mathbb{R}$. Suppose $\alpha < \beta$. Then there exists some $r \in \mathbb{Q}$ such that $\alpha < r < \beta$.*
Remark. It is an exercise on how to cleverly split an argument into various cases.

ii.♢ *Let $\alpha, \beta \in \mathbb{R}$. Suppose $\alpha < \beta$. Then there exists some $r \in \mathbb{R}\backslash\mathbb{Q}$ such that $\alpha < u < \beta$.*

(d) Prove the statements below:

i.♣ *Suppose $x \in \mathbb{R}$. Then there exists some increasing infinite sequence of rational numbers $\{a_n\}_{n=1}^{\infty}$ such that $x - \frac{1}{2n} < a_n < x$ for any $n \in \mathbb{N}\backslash\{0\}$.*
(*Hint.* First apply the Archimedean Principle to obtain some infinite sequence of rational numbers $\{c_n\}_{n=0}^{\infty}$ which satisfies $x - \frac{1}{2n} < c_n < x$ for any $n \in \mathbb{N}\backslash\{0\}$ but which is not necessarily increasing.)

ii.♢ *Suppose $x \in \mathbb{R}$. Then there exists some decreasing infinite sequence of rational numbers $\{b_n\}_{n=1}^{\infty}$ such that $x < b_n < x + \frac{1}{2n}$ for any $n \in \mathbb{N}\backslash\{0\}$.*
(*Hint.* Make clever use of the result in the previous part.)

iii.♣ *Suppose $x \in \mathbb{R}$. Then there exist some infinite sequence of closed and bounded intervals $\{I_n\}_{n=1}^{\infty}$ such that Conditions* (N1), (N2), (N3), (N4) *are all satisfied:*

(N1) *$x \in I_n$ for any $x \in \mathbb{N}\backslash\{0\}$.*

(N2) *For any $n \in \mathbb{N}\backslash\{0\}$, the endpoints of I_n are rational numbers whose distance from each other is most $\frac{1}{n}$, and neither endpoints of I_n is x.*

(N3) *$I_{n+1} \subset I_n$ for any $n \in \mathbb{N}\backslash\{0\}$.*

(N4) *$\{u \in \mathbb{R} : u \in I_n$ for any $n \in \mathbb{N}\backslash\{0\}\} = \{x\}$.*

(*Hint.* Apply the results in the two parts above.)