MATH1050 Exercise 8 (Answers and selected solution)

1. —

2. **Solution.**

   Let r be a positive rational number. For any $n \in \mathbb{N}\backslash\{0\}$, define $a_n = \sum_{k=1}^{n} \frac{1}{k^{1+r}}$.

   (a) Pick any $n \in \mathbb{N}\backslash\{0\}$. We have $a_{n+1} - a_n = \frac{1}{(n+1)^{1+r}} > 0$. Then $a_{n+1} > a_n$.

   It follows that $\{a_n\}_{n=1}^{\infty}$ is strictly increasing.

   (b) i. For each $m \geq 1$, we have

$$a_{2^{m+1}-1} - a_{2^m-1} = \sum_{k=1}^{2^{m+1}-1} \frac{1}{k^{1+r}} - \sum_{k=1}^{2^m-1} \frac{1}{k^{1+r}} = \sum_{k=2^m}^{2^{m+1}-1} \frac{1}{k^{1+r}} = \sum_{k=2^m}^{2^{m+1}-1} \frac{1}{k \cdot k^r}$$

$$\leq \sum_{k=2^m}^{2^{m+1}-1} \frac{1}{2^m \cdot 2^{mr}} = \frac{2^m}{2^m \cdot 2^{mr}} = \frac{1}{2^{mr}}$$

   ii. Pick any $n \in \mathbb{N}\backslash\{0\}$. We have $n \leq 2^n - 1$. Then

$$\begin{aligned} a_n \leq a_{2^n-1} = \sum_{k=1}^{2^n-1} \frac{1}{k^{1+r}} = a_1 + \sum_{j=1}^{n-1}(a_{2^{j+1}-1} - a_{2^j-1}) &\leq 1 + \sum_{j=1}^{n-1} \frac{1}{2^{jr}} \\ &= \sum_{j=0}^{n-1} \frac{1}{2^{jr}} \\ &= \frac{1 - 1/2^{nr}}{1 - 1/2^r} \\ &\leq \frac{1}{1 - 1/2^r} = \frac{2^r}{2^r - 1} \end{aligned}$$

   Hence $\{a_n\}_{n=1}^{\infty}$ is bounded above by $\frac{2^r}{2^r - 1}$.

3. **Answer.**

   (a) Least element: -1.
   Greatest element: *None.*
   The set concerned is bounded above by 1 in $\mathbb{R}$. (Every real number no less than 1 is an upper bound.)

   (b) Least element: *None.*
   The set concerned is bounded below by -1 in $\mathbb{R}$. (Every real number no greater than -1 is a lower bound.)
   Greatest element: 1.
   The set concerned is bounded above by 1 in $\mathbb{R}$. (Every real number no less than 1 ia an upper bound.)

   (c) Least element: *None.*
   The set concerned is bounded below by 0 in $\mathbb{R}$. (Every real number no greater than 0 is a lower bound.)
   Greatest element: 1.

   (d) Least element: *None.*
   The set concerned is bounded below by -1 in $\mathbb{R}$. (Every real number no greater than -1 is a lower bound.)
   Greatest element: 2.

   (e) Least element: *None.*
   The set concerned is bounded below by 1 in $\mathbb{R}$. (Every real number no greater than 1 is a lower bound.)
   Greatest element: *None.*
   The set concerned is not bounded above in $\mathbb{R}$.

   (f) Least element: *None.*
   The set concerned is bounded below by 1 in $\mathbb{R}$. (Every real number no greater than 1 is a lower bound.)
   Greatest element: *None.*
   The set concerned is not bounded above in $\mathbb{R}$.

   (g) Least element: *None.*
   The set concerned is bounded below by $-\frac{3}{2}$ in $\mathbb{R}$. (Every real number no greater than $-\frac{3}{2}$ is a lower bound.)
   Greatest element: *None*
   The set concerned is not bounded above in $\mathbb{R}$.

   (h) Least element: -1.
   Greatest element: 2.

(i) Least element: *None.*
The set concerned is not bounded below in $\mathbb{R}$.
Greatest element: *None.*
The set concerned is bounded above by 1 in $\mathbb{R}$. (Every real number no less than 1 is a upper bound.)

(j) Least element: *None.*
The set concerned is bounded below by -3 in $\mathbb{R}$. (Every real number no greater than -3 is a lower bound.)
Greatest element: *None.*
The set concerned is bounded above by 1 in $\mathbb{R}$. (Every real number no less than 1 is a upper bound.)

(k) Least element: *None.*
The set concerned is bounded below by 0 in $\mathbb{R}$. (Every real number no greater than 0 is a lower bound.)
Greatest element: 2.

(l) Least element: *None.*
The set concerned is bounded below by -1 in $\mathbb{R}$. (Every real number no greater than -1 is a lower bound.)
Greatest element: *None.*
The set concerned is bounded below by 1 in $\mathbb{R}$. (Every real number no less than 1 is a upper bound.)

4. ——

5. **Answer.**

(a) ——

(b) $\frac{1}{9}$ is the least element of T.

(c) *Hint.* $\frac{1}{27}$ is an element of S and is not an element of T.

(d) *Hint.* Given that $u, v \in S$ and $u < v$, is it true that $\frac{2u+v}{3} \in S$ and $u < \frac{2u+v}{3} < v$?

6. (a) **Answer.**

(I) Suppose $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are non-negative real numbers.

Then $\left(\sum_{j=1}^{n} {a_j}^2\right)\left(\sum_{j=1}^{n} {b_j}^2\right) \geq \left(\sum_{j=1}^{n} a_j b_j\right)^2$.

(II) $(sv - tu)^2$

(III) $(s^2 + t^2)(u^2 + v^2)$

(IV) $(su + tv)^2$

(V) $a_1, a_2, \cdots, a_m, a_{m+1}, b_1, b_2, \cdots, b_m, b_{m+1}$ are non-negative real numbers

(VI) $\sqrt{\sum_{j=1}^{m} a_j b_j}$

(VII) $\left(A^2 + {a_{m+1}}^2\right)\left(B^2 + {b_{m+1}}^2\right)$

(VIII) $a_{m+1} b_{m+1}$

(IX) C^2

(b) **Solution.**

Let $n \in \mathbb{N} \backslash \{0, 1\}$. Suppose $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are real numbers.
Note that $|a_1|, |a_2|, \cdots, |a_n|, |b_1|, |b_2|, \cdots, |b_n|$ are non-negative real numbers.

By the result in part (a), we have $\left(\sum_{j=1}^{n} |a_j|^2\right)\left(\sum_{j=1}^{n} |b_j|^2\right) \geq \left(\sum_{j=1}^{n} |a_j b_j|\right)^2$.

Note that $\left(\sum_{j=1}^{n} |a_j|^2\right) = \left(\sum_{j=1}^{n} {a_j}^2\right)$, $\left(\sum_{j=1}^{n} |b_j|^2\right) = \left(\sum_{j=1}^{n} {b_j}^2\right)$.

By the Triangle Inequality for real numbers, we have $\sum_{j=1}^{n} |a_j b_j| \geq \left|\sum_{j=1}^{n} a_j b_j\right|$.

Then $\left(\sum_{j=1}^{n} |a_j b_j|\right)^2 \geq \left|\sum_{j=1}^{n} a_j b_j\right|^2 = \left(\sum_{j=1}^{n} a_j b_j\right)^2$.

Therefore $\left(\sum_{j=1}^{n} {a_j}^2\right)\left(\sum_{j=1}^{n} {b_j}^2\right) = \left(\sum_{j=1}^{n} |a_j|^2\right)\left(\sum_{j=1}^{n} |b_j|^2\right) \geq \left(\sum_{j=1}^{n} |a_j b_j|\right)^2 \geq \left(\sum_{j=1}^{n} a_j b_j\right)^2$.

7. (a) *Hint.* Apply the Cauchy-Schwarz Inequality to obtain $|yz+zx+xy| \leq x^2+y^2+z^2$. Then check whether $(y+z-x)^2+(z+x-y)^2+(x+y-z)^2-(yz+zx+xy)$ is non-negative.

   (b) *Hint.* Define $c=\sqrt{a}$. Apply the Cauchy-Schwarz Inequality to obtain

$$\begin{aligned}(2n+1)^2 \quad &\leq \quad (c^{-2n}+c^{-2(n-1)}+\cdots+c^{-4}+c^{-2}+1+c^2+c^4+\cdots+c^{2(n-1)}+c^{2n}) \\ &\quad\cdot \quad (c^{2n}+c^{2(n-1)}+\cdots+c^4+c^2+1+c^{-2}+c^{-4}+\cdots+c^{-2(n-1)}+c^{-2n})\end{aligned}$$

   (c) Apply the Cauchy-Schwarz Inequality to obtain $\left(\sum_{k=0}^{n} 1\cdot\sqrt{\binom{n}{k}}\right)^2 \leq \left(\sum_{k=0}^{n}\binom{n}{k}\right)\left(\sum_{k=0}^{n}1^2\right)$

8. (a) *Hint.* Apply the Cauchy-Schwarz Inequality to obtain $\left(\sum_{k=1}^{n} a_k\cdot 1\right)^2 \leq \left(\sum_{k=1}^{n}1^2\right)\left(\sum_{k=1}^{n}a_k{}^2\right)$.

   (b) ——

   (c) ——

9. (a) *Hint.* Define $\kappa=u^p$, $\lambda=v^q$, and $\gamma=\left(\frac{\kappa}{\lambda}\right)^{1/p}-1$.

   Apply Bernoulli's Inequality to obtain $(1+\gamma)^p \geq 1+p\gamma$. (Why applicable? Check the conditions first.)

   (b) *Hint.* Define $H=(a^p+b^p)^{1/p}$, $K=(c^q+d^q)^{1/q}$, and $\alpha=a/H$, $\beta=b/H$, $\gamma=c/K$, $\delta=d/K$.

   Apply the result in part (a) to obtain $\alpha\gamma \leq \frac{\alpha^p}{p}+\frac{\gamma^q}{q}$ and $\beta\delta \leq \frac{\beta^p}{p}+\frac{\delta^q}{q}$. (Why applicable? Check the conditions first.)

   (c) *Hint.* Apply the result in part (b) to obtain

$$\begin{aligned}w(w+y)^{p-1}+x(x+z)^{p-1} \quad &\leq \quad (w^p+x^p)^{1/p}\left[(w+y)^p+(x+z)^p\right]^{1/q}, \\ y(w+y)^{p-1}+z(x+z)^{p-1} \quad &\leq \quad (y^p+z^p)^{1/p}\left[(w+y)^p+(x+z)^p\right]^{1/q}.\end{aligned}$$

   (Why applicable? Check the conditions first.) Now add up the respective 'left-hand-sides' and 'right-hand-sides' of the inequalities and see what happens.

Remark. How about generalizing the arguments in part (b) and part (c) to prove the statements below?

- *Let $p\in(1,+\infty)\cap\mathbb{Q}$. Define $q=\left(1-\frac{1}{p}\right)^{-1}$. Let $a_1,a_2,\cdots,a_n,c_1,c_2,\cdots,c_n$ be positive real numbers.*

  The inequality $\sum_{j=1}^{n}a_jc_j \leq \left(\sum_{j=1}^{n}a_j{}^p\right)^{\frac{1}{p}}\left(\sum_{j=1}^{n}c_j{}^p\right)^{\frac{1}{q}}$ holds.

- *Let $p\in(1,+\infty)\cap\mathbb{Q}$. Let $w_1,w_2,\cdots,w_n,y_1,y_2,\cdots,y_n$ be positive real numbers.*

  The inequality $\left[\sum_{j=1}^{n}(w_j+y_j)^p\right]^{\frac{1}{p}} \leq \left(\sum_{j=1}^{n}w_j{}^p\right)^{\frac{1}{p}}+\left(\sum_{j=1}^{n}y_j{}^p\right)^{\frac{1}{p}}$ holds.

10. (a) ——

    (b) *Hint.* First express the 'left-hand-side' as a fraction, and simplify the expression using the assumption $a+b+c=1$. Then apply the Arithmetico-geometrical Inequality, possibly more than once.

    (c) *Hint.* Refer to part (i). Look at the 'left-hand-side' of the inequality to be deduced. How are the individual 'factors' in the product related to each other? How does this product related to the sum (instead of product) of the individual 'factors'?

11. (a) *Hint.* This inequality is Bernoulli's Inequality in disguise.

    (b) *Hint.* Define $t=\frac{G_{k+1}}{G_k}$, and $n=k+1$. Apply the result in part (a).

    (c) ——

12. ——

13. ——

14. (a) **Answer.**
    AGDCJHIFEB.
    Alternative answers. **GADCJHIFEB**.

(b) **Solution.**
Pick any $\varepsilon > 0$.
Note that $\frac{1}{\varepsilon} > 0$. By ($\flat$), the number $\frac{1}{\varepsilon}$ is not an upper bound of $\mathbb{N}$ in $\mathbb{R}$.
Then there exists some $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. By definition, $N \in \mathbb{N}\backslash\{0\}$ and $N\varepsilon > 1$.

(c) **Solution.**
Suppose $x, u \in (0, +\infty)$.

- (Existence argument.) By the Archimedean Principle, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N \cdot \frac{u}{x} > 1$. For the same N, we have $Nu > x$.
Define $S = \{k \in \mathbb{N} : ku > x\}$. Note that $N \in S$. Then $S \neq \emptyset$. By the Well-ordering Principle for integers, S has a least element. Denote it by ν.
Since $\nu u > x > 0$, we have $\nu > 0$. Then $\nu - 1 \in \mathbb{N}$.
Define $q = \nu - 1$, $r = x - qu$.
By definition, $x = qu + r$ and $q \in \mathbb{N}$ and $r \in \mathbb{R}$.
Since ν is a least element of S and $q = \nu - 1 < \nu$, we have $q \notin S$. Then $x \geq qu$. Therefore $r = x - qu \geq 0$.
We verify that $r \in [0, u)$:
  * Suppose it were true that $r < 0$. By definition, $x - qu = r < 0$. Then $x < qu$. Since $q \in \mathbb{N}$, we would have $q \in S$. But $q = \nu - 1 < \nu$, and ν is a least element of S. Contradiction arises.
  Hence $r \geq 0$ in the first place.
  * Suppose it were true that $r \geq u$. Define $\hat{r} = r - u$. By definition, $\hat{r} \geq 0$.
  Now we would have $0 \geq \hat{r} = r - u = x - qu - u = x - (q+1)u$. Then $x \geq (q+1)u = \nu u$. We would have $\nu \notin S$. But $\nu \in S$. Contradiction arises.
  Hence $r < u$ in the first place.
- (Uniqueness argument.) Suppose $q, q' \in \mathbb{N}$, and $r, r' \in [0, u)$. Suppose $x = qu + r$ and $x = q'u + r'$
We have $qu + r = q'u + r'$. Therefore $|q - q'|u = |r' - r|$. Since $0 \leq r < u$ and $0 \leq r' < u$, we have $0 \leq |q - q'|u = |r - r'| < u$. Now note that $u \in (0, +\infty)$ and $|q - q'| \in \mathbb{N}$. Therefore $|q - q'|u = 0$ or $|q - q'|u \geq u$. Since $|q - q'|u < u$, we have $|q - q'|u = 0$. Therefore $q = q'$. Hence $r = r'$ also.