MATH1050 Exercise 8

1. Consider each of the infinite sequences (of non-negative real numbers) below. Verify that it is strictly monotonic. Also name an upper bound and a lower bound for each of them.

   (a) $\left\{\dfrac{1}{n}\right\}_{n=1}^{\infty}$

   (b) $\left\{\dfrac{1}{n^2}\right\}_{n=1}^{\infty}$

   (c) $\left\{\dfrac{n+1}{n^2+n+1}\right\}_{n=0}^{\infty}$

   (d) $\left\{\dfrac{2}{3^n}\right\}_{n=0}^{\infty}$

   (e) $\left\{\dfrac{5^n}{n!}\right\}_{n=5}^{\infty}$

   (f) $\left\{\dfrac{(n!)^2}{(2n)!}\right\}_{n=0}^{\infty}$

   (g) $\left\{\sqrt[n]{2}\right\}_{n=2}^{\infty}$

   (h) $\left\{\displaystyle\sum_{k=0}^{n}\frac{1}{3^k}\right\}_{n=0}^{\infty}$

   (i)$^\diamondsuit$ $\left\{\displaystyle\sum_{k=2}^{n}\frac{2}{k^2-1}\right\}_{n=2}^{\infty}$

   (j) $\left\{\dfrac{n^2}{3^n}\right\}_{n=2}^{\infty}$

2. Let r be a positive rational number. For any $n \in \mathbb{N}\backslash\{0\}$, define $a_n = \displaystyle\sum_{k=1}^{n}\frac{1}{k^{1+r}}$.

   (a) Verify that $\{a_n\}_{n=1}^{\infty}$ is strictly increasing.

   (b) i. Prove that for any $m \in \mathbb{N}\backslash\{0\}$, $a_{2^{m+1}-1} - a_{2^m-1} \leq \dfrac{1}{2^{mr}}$

   ii.$^\diamondsuit$ Hence deduce that $\{a_n\}_{n=1}^{\infty}$ is bounded above in $\mathbb{R}$. (*Hint.* Apply the Telescopic Method.)

   Remark. It follows from the Bounded-Monotone Theorem that, for each rational number σ greater than 1, the infinite sequence $\left\{\displaystyle\sum_{k=1}^{n}\frac{1}{k^\sigma}\right\}_{n=1}^{\infty}$ converges in $\mathbb{R}$. In fact, for each real number $x > 1$, the infinite sequence $\left\{\displaystyle\sum_{k=1}^{n}\frac{1}{k^x}\right\}_{n=1}^{\infty}$ converges in $\mathbb{R}$. The function defined by assigning each $x \in (1, +\infty)$ to the number $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^x}$, and then 'extended naturally to all complex numbers with real part greater than 1', is the known as the **Riemann Zeta-Function**.

3. Consider the subsets of $\mathbb{R}$ below.

   - Determine whether it has any least element. If *yes*, name it as well. If it has no least element, determine whether it has a lower bound in $\mathbb{R}$.
   - Determine whether it has any greatest element. If *yes*, name it as well. If it has no greatest element, determine whether it has an upper bound in $\mathbb{R}$.

   There is no need to justify your answers. (Drawing appropriate pictures, on the real line or on the coordinate plane, may help you find the answers.)

   (a) $[-1, 1) \cap \mathbb{Q}$

   (b) $[-1, 1)\backslash\mathbb{Q}$

   (c) $\left\{\dfrac{1}{n+1}\,\middle|\, n \in \mathbb{N}.\right\}$

   (d)$^\diamondsuit$ $\left\{\dfrac{1}{n+1}+(-1)^n\,\middle|\, n\in\mathbb{N}.\right\}$

   (e) $(1, +\infty) \cap \mathbb{Q}$

   (f) $(1, +\infty)\backslash\mathbb{Q}$

   (g) $\{x \in \mathbb{R} : 2x + 3 > 0.\}$

   (h) $\{x \in \mathbb{R} : x + 2 \geq x^2.\}$

   (i) $\{x \in \mathbb{R} : x < x^{-1}.\}$

   (j) $\{x \in \mathbb{R} : x^2-2x-3<0.\}$

   (k)$^\diamondsuit$ $\left\{\dfrac{1}{2^m}+\dfrac{1}{3^n}\,\middle|\, m, n \in \mathbb{N}.\right\}$

   (l)$^\clubsuit$ $\left\{\dfrac{1}{2^m}-\dfrac{1}{3^n}\,\middle|\, m, n \in \mathbb{N}.\right\}$

4. (a) Let $I = [0, 9)$. (By definition, $I = \{x \in \mathbb{R} : 0 \leq x < 9\}$.)

   i. Prove that I has a least element, namely, 0.

   ii.$^\diamondsuit$ Prove that I has no greatest element.

   (b) Let $J = [0, 9) \cap \mathbb{Q}$.

   i. Prove that J has a least element, namely, 0.

   ii.$^\diamondsuit$ Prove that J has no greatest element.

   (c) Let $K = [0, 9)\backslash\mathbb{Q}$.

   i.$^\diamondsuit$ Prove that K has no greatest element.

   ii.$^\clubsuit$ Prove that K has no least element.

5. Let $S = \left\{x \in \left(0, \dfrac{1}{8}\right) : x = \dfrac{b}{3^a} \text{ for some } a, b \in \mathbb{N}\right\}$, and $T = \left\{y \in \mathbb{R} : y = \displaystyle\sum_{k=1}^{n}\frac{1}{9^k} \text{ for some } n \in \mathbb{N}\backslash\{0\}\right\}$.

   (a) Verify that $T \subset S$.

   (b) Does T have a least element? Justify your answer.

   (c) Prove that $S \not\subset T$.

   Remark. The result you obtain in part (b) may be useful.

   (d)$^\diamondsuit$ Prove the statement below:

   - *For any $u, v \in S$, if $u < v$ then there exists some $w \in S$ such that $u < w < v$.*

6. *Here we are going to re-prove of the 'inequality part' of Cauchy-Schwarz Inequality, with the help of mathematical induction and the Triangle Inequality for the reals.*

(a) Consider the statement (S):

Let $n \in \mathbb{N}\backslash\{0,1\}$. Suppose $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are non-negative real numbers.

Then $\left(\sum_{j=1}^{n} {a_j}^2\right)\left(\sum_{j=1}^{n} {b_j}^2\right) \geq \left(\sum_{j=1}^{n} a_j b_j\right)^2$.

Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate passages so that it gives an argument for the statement (S) by mathematical induction.

Denote by $P(n)$ the proposition below:

__________ (I) __________

- Suppose s, t, u, v are non-negative real numbers.

  We have $(s^2+t^2)(u^2+v^2)-(su+tv)^2 =$ ____ (II) ____ ≥ 0.

  Then ____ (III) ____ $\geq$ ____ (IV) ____ .

  Hence $P(2)$ is true.

- Let $m \in \mathbb{N}\backslash\{0,1\}$. Suppose $P(m)$ is true. We verify that $P(m+1)$ is true below:

  Suppose ________ (V) ________ .

  Define $A = \sqrt{\sum_{j=1}^{m} {a_j}^2}$, $B = \sqrt{\sum_{j=1}^{m} {b_j}^2}$, $C =$ ____ (VI) ____ .

  Note that A, B, C are non-negative real numbers.

  By $P(2)$, we have $\left(\sum_{j=1}^{m+1} {a_j}^2\right)\left(\sum_{j=1}^{m+1} {b_j}^2\right) =$ ____ (VII) ____ $\geq (AB + a_{m+1}b_{m+1})^2$.

  By $P(m)$, we have $AB \geq C^2$. Then $AB +$ ____ (VIII) ____ $\geq$ ____ (IX) ____ $+ a_{m+1}b_{m+1} = \sum_{j=1}^{m+1} a_j b_j \geq 0$.

  Therefore $\left(\sum_{j=1}^{m+1} {a_j}^2\right)\left(\sum_{j=1}^{m+1} {b_j}^2\right) \geq (AB + a_{m+1}b_{m+1})^2 \geq \left(\sum_{j=1}^{m+1} a_j b_j\right)^2$.

  Hence $P(m+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}\backslash\{0,1\}$.

(b) By applying the result above together with the Triangle Inequality for the reals, or otherwise, prove the statement below:

Let $n \in \mathbb{N}\backslash\{0,1\}$. Suppose $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are real numbers.

Then $\left(\sum_{j=1}^{n} {a_j}^2\right)\left(\sum_{j=1}^{n} {b_j}^2\right) \geq \left(\sum_{j=1}^{n} a_j b_j\right)^2$.

7. *The various parts in this question are concerned with applications of the Cauchy-Schwarz Inequality. They are independent of each other.*

(a) Suppose x, y, z are real numbers. Prove that $yz + zx + xy \leq (y+z-x)^2 + (z+x-y)^2 + (x+y-z)^2$.

(b) Let $a > 0$. Prove that $\dfrac{a^n}{1+a+a^2+\cdots+a^{2n}} \leq \dfrac{1}{2n+1}$.

(c) Let n be a positive integer. Prove that $\sum_{k=0}^{n} \sqrt{\binom{n}{k}} \leq \sqrt{2^n(n+1)}$.

8. (a) Let $a_1, a_2, \cdots, a_n$ be positive real numbers. Prove that $\dfrac{1}{n}\sum_{k=1}^{n} a_k \leq \sqrt{\dfrac{1}{n}\sum_{k=1}^{n} {a_k}^2}$.

(b)$^\diamondsuit$ Let $b_1, b_2, \cdots, b_n$ be positive real numbers. Suppose $\sum_{k=1}^{n} b_k = S$. Prove that $\sum_{k=1}^{n} \sqrt{b_k} \leq \sqrt{nS}$.

(c)$^\diamondsuit$ Let $c_1, c_2, \cdots, c_n$ be positive real numbers. Suppose $\sum_{k=1}^{n} c_k = 1 + \frac{1}{2n}$.

By applying the previous part, or otherwise, prove that $\sum_{k=1}^{n} \sqrt{2c_k + 1} \leq n + 1$.

9.$^\clubsuit$ *In this question you may need this version of* **Bernoulli's Inequality***:*

- *Let $m \in (1, +\infty) \cap \mathbb{Q}$ and $\beta \in (-1, +\infty)$. The inequality $(1+\beta)^m \geq 1 + m\beta$ holds. Equality holds iff $\beta = 0$.*

Let $p \in (1, +\infty) \cap \mathbb{Q}$. Define $q = \left(1 - \frac{1}{p}\right)^{-1}$. (Note that $q \in (1, +\infty) \cap \mathbb{Q}$ and $\frac{1}{p} + \frac{1}{q} = 1$.)

Prove the results below:

(a) *Let u, v be positive real numbers. The inequality $uv \leq \frac{u^p}{p} + \frac{v^q}{q}$ holds.*

(b) *Let a, b, c, d be positive real numbers. The inequality $ac + bd \leq (a^p + b^p)^{\frac{1}{p}} (c^q + d^q)^{\frac{1}{q}}$ holds.*

(c) *Let w, x, y, z be positive real numbers. The inequality $[(w+y)^p + (x+z)^p]^{\frac{1}{p}} \leq (w^p + x^p)^{\frac{1}{p}} + (y^p + z^p)^{\frac{1}{p}}$ holds.*

Remark. Apply Bernoulli's Inequality in part (a). In part (b), apply the result of part (a). In part (c), apply the result of part (b). The results in part (b), part (c) are 'baby versions' of **Hölder's Inequality**, **Minkowski's Inequality** respectively.

10. *The various parts in this question are concerned with applications of the Arithmetico-geometrical Inequality. They are independent of each other.*

(a) Suppose w, x, y, z are real numbers. Prove that $\frac{w^4 + x^2y^2 + y^2z^2 + z^2x^2}{4} \geq wxyz$.

(b) Let a, b, c be positive real numbers. Suppose $a + b + c = 1$. Prove that $\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right) \geq 8$.

(c) Let n be a positive integer.

i. Prove that $n^n \geq 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3) \cdot (2n-1)$.

ii. Hence deduce that $(n^2 + n)^n \geq (2n)!$.

11. *Here we are going to re-prove the Arithmetico-geometrical Inequality.*

(a) Let n be a positive integer, and t be a positive real number. Prove that $t^n - 1 \geq n(t-1)$.

(b)$^\clubsuit$ Let $x_1, x_2, \cdots, x_k, x_{k+1}$ be positive real numbers. Define $G_k = \sqrt[k]{x_1 x_2 \cdot \ldots \cdot x_k}$, $G_{k+1} = \sqrt[k+1]{x_1 x_2 \cdot \ldots \cdot x_k x_{k+1}}$. Prove that $x_{k+1} \geq (k+1)G_{k+1} - kG_k$.

(c) Suppose $\{a_n\}_{n=1}^{\infty}$ is an infinite sequence of positive real numbers.

i.$^\clubsuit$ By applying the result in the previous part, or otherwise, prove that for each integer $m \geq 2$,

$$\frac{a_1 + a_2 + \cdots + a_{m-1} + a_m}{m} - (a_1 a_2 \cdot \ldots \cdot a_{m-1} a_m)^{\frac{1}{m}} \geq \frac{m-1}{m} \left[\frac{a_1 + a_2 + \cdots + a_{m-1}}{m-1} - (a_1 a_2 \cdot \ldots \cdot a_{m-1})^{\frac{1}{m-1}}\right].$$

ii.$^\diamondsuit$ Hence deduce that $\frac{a_1 + a_2 + \cdots + a_{m-1} + a_m}{m} \geq (a_1 a_2 \cdot \ldots \cdot a_{m-1} a_m)^{\frac{1}{m}}$ for each integer $m \geq 2$.

12. (a) Let $x, y \in (0, 0.5]$. Prove that $\frac{xy}{(x+y)^2} \leq \frac{(1-x)(1-y)}{[(1-x)+(1-y)]^2}$.

(b)$^\diamondsuit$ Apply mathematical induction to justify the statement (↑) below:

(↑) *Let $n \in \mathbb{N}$. Suppose $a_1, a_2, \cdots, a_{2^n} \in (0, 0.5]$. Then*

$$\frac{a_1 a_2 \cdot \ldots \cdot a_{2^n}}{(a_1 + a_2 + \cdots + a_{2^n})^{2^n}} \leq \frac{(1-a_1)(1-a_2) \cdot \ldots \cdot (1-a_{2^n})}{[(1-a_1) + (1-a_2) + \cdots + (1-a_{2^n})]^{2^n}}$$

.

(c)$^\clubsuit$ Hence, or otherwise, prove that the statement (⇑) below is true:

(⇑) *Let $n \in \mathbb{N} \backslash \{0, 1\}$. Suppose $a_1, a_2, \cdots, a_n \in (0, 0.5]$. Then*

$$\frac{a_1 a_2 \cdot \ldots \cdot a_n}{(a_1 + a_2 + \cdots + a_n)^n} \leq \frac{(1-a_1)(1-a_2) \cdot \ldots \cdot (1-a_n)}{[(1-a_1) + (1-a_2) + \cdots + (1-a_n)]^n}$$

Remark. Equality holds iff $a_1 = a_2 = \cdots = a_n$. (Prove it as well.) The result (together with the 'equality condition') that we have proved is known as the **Ky Fan Inequality**.

13. (a) Applying the Arithmetico-geometrical Inequality, or otherwise, prove the statement (†) below:

($\dagger$) *Suppose a, b be distinct positive real numbers. Then, for each $n \in \mathbb{N}$, the inequality* $\sqrt[n+1]{ab^n} < \dfrac{a+nb}{n+1}$ *holds.*

(b) For any $n \in \mathbb{N}\backslash\{0,1,2\}$, define $p_n = \left(1+\dfrac{1}{n}\right)^n$, $q_n = \left(1-\dfrac{1}{n}\right)^n$, $r_n = \left(1+\dfrac{1}{n}\right)^{n+1}$

i. Applying the statement ($\dagger$), or otherwise, prove that $\{p_n\}_{n=2}^{\infty}$, $\{q_n\}_{n=2}^{\infty}$ are strictly increasing.

ii. Hence, or otherwise, prove that $\{r_n\}_{n=2}^{\infty}$ is strictly decreasing.
(*Hint.* Can you relate r_n with one of q_{n-1}, q_n, q_{n+1}? And which of them?)

iii. Hence, or otherwise, deduce that $\{p_n\}_{n=2}^{\infty}$ is bounded above in $\mathbb{R}$ and $\{r_n\}_{n=2}^{\infty}$ is bounded below in $\mathbb{R}$.

(c) *Now, according to the Bounded-Monotone Theorem, the infinite sequence $\{p_n\}_{n=2}^{\infty}$ converges to some limit in $\mathbb{R}$, which we denote by e. Moreover, (after some work), we can deduce that the statement ($\ddagger$) holds:*

($\ddagger$) $\left(1+\dfrac{1}{n}\right)^n < e < \left(1+\dfrac{1}{m}\right)^{m+1}$ *for any $m, n \in \mathbb{N}\backslash\{0,1\}$.*

Applying the statement ($\ddagger$), or otherwise, prove that $\left(\dfrac{n}{e}\right)^n < n! < e\left(\dfrac{n+1}{e}\right)^{n+1}$ for each $n \in \mathbb{N}\backslash\{0,1\}$.

14. (a) Consider the statements ($\sharp$), ($\flat$):

($\sharp$) *The set $\mathbb{N}$ is not bounded above in $\mathbb{R}$.*

($\flat$) *Let A be a non-empty subset of $\mathbb{R}$. Suppose A is bounded above in $\mathbb{R}$. Then the set of all upper bounds of A in $\mathbb{R}$ has a least element.*

The validity of the statement ($\flat$), which is the **Least-upper-bound Axiom**, *is taken for granted.*

By an appropriate re-ordering of the blocks of sentences in the box below, labelled by bold-typed Latin alphabets **A**, **B**, ..., **J** respectively, give a proof for the statement ($\sharp$), with the help of the statement ($\flat$).

A. *Note that $0 \in \mathbb{N}$. Then $\mathbb{N} \neq \emptyset$.*

B. *Contradiction arises. Hence $\mathbb{N}$ is not bounded above in $\mathbb{R}$ in the first place.*

C. *Write $\varepsilon_0 = \dfrac{1}{2}$. We have $\sigma - \varepsilon_0 < \sigma$. By definition of least element, $\sigma - \varepsilon_0 \notin B$.*

D. *Then, by ($\flat$), B would have a least element. We denote this number by σ.*

E. *Hence $\sigma \in B$ and $\sigma \notin B$ simultaneously.*

F. *Then, because $n_0 + 1 > \sigma$ and $n_0 + 1 \in \mathbb{N}$, σ would not be an upper bound of $\mathbb{N}$ in $\mathbb{R}$. Therefore $\sigma \notin B$.*

G. *Suppose it were true that $\mathbb{N}$ was bounded above in $\mathbb{R}$.
Define $B = \{\tau \in \mathbb{R} : \tau$ is an upper bound of $\mathbb{N}$ in $\mathbb{R}\}$.*

H. *Therefore there would exist some $n_0 \in \mathbb{N}$ such that $n_0 > \sigma - \varepsilon_0$. Since $n_0 \in \mathbb{N}$, we have $n_0 + 1 \in \mathbb{N}$.*

I. *Note that $n_0 + 1 > \sigma - \varepsilon_0 + 1 = \sigma + \dfrac{1}{2} > \sigma$.*

J. *By the definition of B, the number $\sigma - \varepsilon_0$ is not an upper bound of $\mathbb{N}$.*

(b) Apply the statement ($\sharp$) to deduce the statement below, known as the **Archimedean Principle for the reals**:

(AP) *For any $\varepsilon > 0$, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N\varepsilon > 1$.*

Remark. The Archimedean Principle can be (trivially) re-formulated as:

(AP') *For any $\varepsilon > 0$, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $\dfrac{1}{N} < \varepsilon$.*

It is in fact logically equivalent to the statement ($\sharp$) given in the previous part. In various textbooks in *mathematical analysis*, the name 'Archimedean Principle' may refer to any one of these three logically equivalent statements.

(c)♣ Apply the Archimedean Principle and the Well-ordering Principle for integers to prove the statement below:

- *Suppose $x, u \in (0, +\infty)$. Then there exist some unique $q \in \mathbb{N}$, $r \in [0, u)$ such that $x = qu + r$.*

(*Hint.* Consider the set $\{k \in \mathbb{N} : ku > x\}$. Is it true that it has a least element? Can it be justified with the Well-ordering Principle for integers? But is this set non-empty in the first place? Can it be justified with the Archimedean Principle. If this set has a least element, say, ν, what can be said of the number $\nu - 1$? Is the latter a natural number? Now what about the number $x - (\nu - 1)u$? Is it true that $x - (\nu - 1)u$ is non-negative and less than u?)

Remark. Can you provide a geometric interpretation for this result when you are given a pair of line segments of length x, u respectively?