MATH1050 Exercise 7 Supplement (Answers)

1. (a) There exists some $\varepsilon \in (0, +\infty)$ such that for any $\delta \in (0, +\infty)$, there exists some $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$ and $|f(x) - \ell| \geq \varepsilon$.
   (b) For any $\ell \in \mathbb{R}$, there exists some $\varepsilon \in (0, +\infty)$ such that for any $\delta \in (0, +\infty)$, there exists some $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$ and $|f(x) - \ell| \geq \varepsilon$.
   (c) There exists some $a \in \mathbb{R}$ such that for any $\ell \in \mathbb{R}$, there exists some $\varepsilon \in (0, +\infty)$ such that for any $\delta \in (0, +\infty)$ there exists some $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$ and $|f(x) - \ell| \geq \varepsilon$.
   (d) There exist some $a \in \mathbb{R}$, $\varepsilon \in (0, +\infty)$ such that for any $\delta \in (0, +\infty)$, there exists some $x \in \mathbb{R}$ such that $|x - a| < \delta$ and $|f(x) - f(a)| \geq \varepsilon$.
   (e) There exists some $\varepsilon \in (0, +\infty)$ such that for any $\delta \in (0, +\infty)$ there exists some $x, a \in \mathbb{R}$ such that $|x - a| < \delta$ and $|f(x) - f(a)| \geq \varepsilon$.
   (f) There exists some $\varepsilon \in (0, +\infty)$ such that for any $\delta \in (0, +\infty)$, there exist some $x, a \in \mathbb{R}$ such that $|x - a| < \delta$ and $|f(x) - f(a)| \geq \varepsilon$.

   Remark. The given statement to be negated is:

   '*For any $a \in \mathbb{R}$, for any $\varepsilon \in (0, +\infty)$, there exists some $\delta \in (0, +\infty)$ independent of the choice of a such that for any $x \in \mathbb{R}$, if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.*'

   In light of the presence of the words '*independent of the choice of a*' appending 'there exists some δ', the given statement should be understood as:

   '*For any $\varepsilon \in (0, +\infty)$, there exists some $\delta \in (0, +\infty)$ such that for any $a \in \mathbb{R}$, for any $x \in \mathbb{R}$, if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.*'

2. (a) False. (b) True. (c) True. (d) True. (e) False. (f) False. (g) False. (h) False. (i) False. (j) False. (k) True.

3. ——

4. ——

5. (a) ——
   (b) ——
   (c) i. ——
   ii. A. 4. B. 4.

6. ——

7. ——

8.

9. (a) True.
   (b) True.
   (c) False.
   (d) True.
   (e) False.
   (f) True.
   (g) False.
   (h) False.
   (i) True.
   (j) True.
   (k) True.
   Hint. Start with a concrete prime number, say, 5. Ask: Is $(\sqrt{5})^{\sqrt{5}}$ rational or irrational? If it is irrational, how about $[(\sqrt{5})^{\sqrt{5}}]^{\sqrt{5}}$?
   (l) False.
   (m) False.
   (n) False.
   (o) False.
   (p) False.
   (q) False.
   (r) True.
   (s) False.
   (t) True.

10. *Hint.*
    Note that $|a| + |b| \in S$. (Why?) Hence $S \neq \emptyset$. By the Well-ordering Principle for Integers, S has a least element. Denote it by d. Now verify that $d = \gcd(a, b)$. Note that by definition, $d \in I$.

11. ——

12. ——

13. (a) ——
   (b) ——
   (c) *Hint.*
   Apply the result in part (b).
   (d) ——
   (e) *Hint.*
   Apply the result in part (d).
   (f) ——

14. (a) *Hint.*
   Apply mathematical induction to the proposition $P(n)$ below:
   For any $m \in [\![2, n]\!]$, n is a prime number or a product of at least two prime numbers.
   (b) *Hint.*
   Apply mathematical induction to the proposition $P(n)$ below:
   For any $m \in [\![2, n]\!]$, if $p_1, p_2, \cdots, p_s$, $q_1, q_2, \cdots, q_t$ are prime numbers, $0 < p_1 \leq p_2 \leq \cdots \leq p_s$, $0 < q_1 \leq q_2 \leq \cdots \leq q_t$, $m = p_1 p_2 \cdot ... \cdot p_s$ and $m = q_1 q_2 \cdot ... \cdot q_t$ then $s = t$ and $p_1 = q_1$, $p_2 = q_2$, ... $p_s = q_s$.

15. ——

16. ——

17. (a) ——
   (b) i. True. ii. False. iii. False.

18. (a) True. (b) False. (c) False. (d) True. (e) False. (f) True. (g) False. (h) False.

19. (a) i. True. ii. False.
   (b) ——

20. ——

21. ——

22. ——

23. ——

24. (a) *Hint.* Apply the Triangle Inequality for complex numbers.
   (b) ——
   (c) ——
   (d) ——
   (e) i. ——
   ii. A. $\emptyset$.
   B. $\mathbb{C}$.
   C. $\{\zeta \in \mathbb{C} : |\zeta| < 1\}$.
   D. $\emptyset$.
   E. $\{\zeta \in \mathbb{C} : \mathsf{Re}(\zeta) > 0\}$.
   F. $\{\zeta \in \mathbb{C} : |\mathsf{Re}(\zeta)| + |\mathsf{Im}(\zeta)| < 1\}$.