MATH1050 Exercise 7 (Answers and selected solution.)

1. **Answer.**

   (a) There exists some $x \in \mathbb{R}$ such that for any $s, t \in \mathbb{Q}$, there exists some $n \in \mathbb{Z}$ such that $s < n < t$ and $|x - n| \leq |t - s|$.

   (b) There exists some $p \in \mathbb{R}$, such that for any $q \in \mathbb{R}$, $n \in \mathbb{N}$, there exist some $s, t \in \mathbb{R}$ such that $|s - t| < |q|$ and $|s^n - t^n| \geq |p|$.

   (c) There exists some $s, t \in \mathbb{Q}$ such that for any $p, q \in \mathbb{R}$, there exists some $n \in \mathbb{Z}$ such that $|s - t| \leq |q|$ and ($t^n > |p|$ or $s^n > |p|$).

   (d) For any $n \in \mathbb{N}$, there exists some $\varepsilon \in (0, +\infty)$ such that for any $\delta \in (0, +\infty)$, there exist some $u, v \in \mathbb{C}$ such that $|u - v| < \delta$ and $|u^n - v^n| \geq \varepsilon$.

   (e) There exist some $p, q \in \mathbb{Z}$ such that for any $s, t \in \mathbb{Z}$, there exist some $m, n \in \mathbb{N}$ such that $|p+q| \geq s^m$ and $|p^n - q| \geq t$ and $|p - q^n| \geq t$.

   (f) There exists some $z \in \mathbb{C}$ such that for any $r \in \mathbb{R}$, there exists some $w \in \mathbb{C}$ such that ($|z - w| \leq r$ and ($z \in \mathbb{R}$ or $|w| \leq r$)).

   (g) There exist some $z, w \in \mathbb{C}$ such that $|z - w| \geq |z + w|$ and (for any $s \in \mathbb{R}$, there exists some $t \in \mathbb{R}$ such that ($|z - s - t| \leq w$ and $|z| \geq 1$)).

   (h) There exist some $\zeta, \alpha, \beta \in \mathbb{C}$ such that (there exist some $s, t \in \mathbb{R}$ such that $\zeta = s\alpha + t\beta$) and (for any $p, q \in \mathbb{R}$, $\zeta \neq p\bar{\alpha} + q\bar{\beta}$.)

2. **Solution.**

   (a) Take $n = 3$. Note that $3 \in \mathbb{N}$. Note that $3 + 2 = 5$, $3 + 4 = 7$. The integers $3, 5, 7$ are prime numbers.

   (b) Take $x = \sqrt{2}$. Note that $x \in \mathbb{R}$. We have $x^2 - 2 = (\sqrt{2})^2 - 2 = 2 - 2 = 0$.

   (c) Take $z_0 = \dfrac{1+i}{\sqrt{2}}$. Note that $z_0 \in \mathbb{C}$.

   Also note that ${z_0}^4 = \left(\dfrac{1+i}{\sqrt{2}}\right)^4 = \dfrac{1 + 4i + 6i^2 + 4i^3 + i^4}{4} = \dfrac{1 + 4i - 6 - 4i + 1}{4} = 1$.

3. **Solution.**

   (a) i. $-2 \in \mathbb{Z}$. $-2 + 1 = -1 < 0$.
   ii. $2 \in \mathbb{Z}$. $2 - 1 = 1 > 0$.

   (b) Suppose it were true that there existed some $x \in \mathbb{Z}$ such that ($x + 1 < 0$ and $x - 1 > 0$). For this x, we would have $x < -1$ and $x > 1$. Then $x < -1 < 1 < x$. Therefore $x \neq x$. Contradiction arises.
   Hence it is false that there exists some $x \in \mathbb{Z}$ such that ($x + 1 < 0$ and $x - 1 > 0$).
   Alterntative argument: The negation of the statement 'there exists some $x \in \mathbb{Z}$ such that ($x+1 < 0$ and $x-1 > 0$)' is given by:

   - For any $x \in \mathbb{Z}$, ($x + 1 \geq 0$ or $x - 1 \leq 0$).

   We give a proof of the latter:

   - Let $x \in \mathbb{Z}$. We have $x \geq 0$ or $x \leq 0$. Where $x \geq 0$, we have $x+1 \geq 1 \geq 0$. Where $x \leq 0$, we have $x-1 \leq -1 \leq 0$.

4. **Solution.**

   (a) Let $z \in \mathbb{C}\backslash\{0\}$. Suppose it were true that $\mathsf{Re}(z) = 0$ and $\mathsf{Im}(z) = 0$. Then $z = \mathsf{Re}(z) + i\mathsf{Im}(z) = 0 + i \cdot 0 = 0$. Contradiction arises. Hence $\mathsf{Re}(z) \neq 0$ or $\mathsf{Im}(z) \neq 0$ in the first place.

   (b) The statement 'for any $z \in \mathbb{C}\backslash\{0\}$, $\mathsf{Re}(z) \neq 0$' is false: we have $i \in \mathbb{C}\backslash\{0\}$ and $\mathsf{Re}(i) = 0$. The statement 'for any $w \in \mathbb{C}\backslash\{0\}$, $\mathsf{Im}(w) \neq 0$' is also false: we have $1 \in \mathbb{C}$ and $\mathsf{Im}(1) = 0$.
   Hence the statement '(for any $z \in \mathbb{C}\backslash\{0\}$, $\mathsf{Re}(z) \neq 0$) or (for any $w \in \mathbb{C}\backslash\{0\}$, $\mathsf{Im}(w) \neq 0$)' is false.

5. (a) **Answer.**

   (I) Suppose
   (II) $\eta = a'\zeta + b'\zeta^2$
   (III) $\zeta \neq 0$
   (IV) $a' - a$
   (V) Suppose it were true that $b \neq b'$
   (VI) $\dfrac{a - a'}{b - b'}$

(VII) $a' = a$

(b) i. ——

Remark. The correct way to start the argument is:

Let r be a real number. Let a, a', b, b' be rational numbers. Suppose $r = a + b\sqrt{2}$ and $r = a' + b'\sqrt{2}$.

ii. ——

Remark. The correct way to start the argument is:

Let p, q be real numbers. Suppose $f(x)$ be the cubic polynomial given by $f(x) = x^3 + px + q$.
Let v be a real number. Let α, β be real numbers. Suppose '$u = \alpha$', '$u = \beta$' are real solutions of the equation $f(u) = v$ with unknown u.

6. (a) **Answer.**

There are many correct answers for (II), (III), ..., (IX) collectively.

(I) There exist some $x, y, z \in \mathbb{Z}$ such that each of xy, xz is divisible by 4 and xyz is not divisible by 8.

(II) $y = z = 1$

(III) 4

(IV) 4

(V) $4 = 1 \cdot 4$ and $1 \in \mathbb{Z}$

(VI) 4

(VII) 4 were divisible by 8

(VIII) $4 = 8k$

(IX) $\frac{1}{2}$

(b) i. ——

ii. ——

Remark. Be aware that the negation of the statement to be dis-proved is:

- *There exist some $x, y, z \in \mathbb{N}$ such that $x - y > 0$ and $y - z > 0$ and $x - y$, $y - z$ are divisible by 5 and $x + y + z$ is divisible by 5.*

iii. ——

Remark. Be aware that the negation of the statement to be dis-proved is:

- *There exist some $x, y \in \mathbb{N}$ such that $\sqrt{x^2 + y^2} \notin \mathbb{N}$.*

iv. ——

Remark. Be aware that the negation of the statement to be dis-proved is:

- *There exist some $s, t \in \mathbb{R}$ such that both of $s + t$, st are rational and both of s, t are irrational.*

v. ——

Remark. Be aware that the negation of the statement to be dis-proved is:

- *There exist some $a, b, c \in \mathbb{N}$ such that ab is divisible by c and $c < a$ and $c < b$ and both of a, b are divisible by c.*

7. (a) **Answer.**

(I) Suppose

(II) $u \in \mathbb{R}\backslash\{-1, 0, 1\}$

(III) $u^6 + v^6 \leq 2v^4$

(IV) $u^6 - 2u^4 + u^2 + v^6 - 2v^4 + v^2 \leq 0$

(V) $u^2(u^2 - 1)^2 = 0$

(VI) $u \in \mathbb{R}\backslash\{-1, 0, 1\}$

i. **Solution.**

We verify that for any $x \in \mathbb{R}$, $x^2 + 2x + 3 \geq 0$:

- Let $x \in \mathbb{R}$. We have $x^2 + 2x + 3 = (x + 1)^2 + 2 \geq 0 + 2 = 2 \geq 0$.

Alternative argument:

Suppose it were true that there existed some $x \in \mathbb{R}$ such that $x^2 + 2x + 3 < 0$. Then we would have $0 \leq 2 = 0 + 2 \leq (x + 1)^2 + 2 = x^2 + 2x + 3 < 0$. Contradiction arises.

Hence, in the first place, it is false that there exists some $x \in \mathbb{R}$ such that $x^2 + 2x + 3 < 0$.

ii. **Solution.**

We verify that for any $x, y \in \mathbb{R}\backslash\{0\}$, $(x + y)^2 \neq x^2 + y^2$:

- Pick any $x, y \in \mathbb{R}\backslash\{0\}$. We have $xy \neq 0$. Then $(x + y)^2 - x^2 - y^2 = 2xy \neq 0$. Then $(x + y)^2 \neq x^2 + y^2$.

Hence it is false that there exist some $x, y \in \mathbb{R}\backslash\{0\}$ such that $(x + y)^2 = x^2 + y^2$.

Alternative argument:

Suppose it were true that there existed some $x, y \in \mathbb{R} \backslash \{0\}$ such that $(x+y)^2 = x^2 + y^2$. Then we would have $2xy = (x+y)^2 - x^2 - y^2 = 0$. Since $x \neq 0$ and $y \neq 0$ and $2 \neq 0$, we have $2xy \neq 0$. Contradiction arises. Hence, in the first place, it is false that there exist some $x, y \in \mathbb{R} \backslash \{0\}$ such that $(x+y)^2 = x^2 + y^2$.

iii. ——

iv. ——

v. ——

8. **Solution.**

(a) Let $s \in \mathbb{Z}$.

- Suppose s is not divisible by 2.
  By Division Algorithm for Integers, there exist some $k, r \in \mathbb{Z}$ such that $s = 2k + r$ and $0 \leq r < 2$.
  Since s is not divisible by 2, we have $r \neq 0$. Then $0 < r < 2$. Therefore $r = 1$.
  Hence $s = 2k + 1$ for the same $k \in \mathbb{Z}$.
- Suppose there exists some $k \in \mathbb{Z}$ such that $s = 2k + 1$.
  We claim that s is not divisible by 2:
  - Suppose it were true that s was divisible by 2. Then there exists some $\ell \in \mathbb{Z}$ such that $s = 2\ell$.
    Now we would have $2\ell = s = 2k + 1$. Then $2(\ell - k) = 1$. Since $k, \ell \in \mathbb{Z}$, we have $\ell - k \in \mathbb{Z}$. Therefore 1 would be divisible by 2. But $0 < 1 < 2$.
    Contradiction arises.
    Hence s is not divisible by 2 in the first place.

(b) Let $a, b \in \mathbb{Z}$.

- Suppose ab is an odd integer.
  We claim that both of a, b are odd integers:
  - Suppose it were true that at least one of a, b, say, a was an even integer. Then there would exist some $k \in \mathbb{Z}$ such that $a = 2k$.
    Now we would have $ab = 2kb$. Since $k, b \in \mathbb{Z}$, we have $kb \in \mathbb{Z}$. Therefore ab would be divisible by 2. Hence ab would not be an odd integer. Contradiction arises.
    Hence both of a, b are odd integers in the first place.
- Suppose both of a, b are odd integers.
  Then there exist $m, n \in \mathbb{Z}$ such that $a = 2m + 1$ and $b = 2n + 1$.
  Now we have $ab = (2m+1)(2n+1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1$.
  Since $m, n \in \mathbb{Z}$, we have $2mn + m + n \in \mathbb{Z}$. Then ab is an odd integer.

9. ——

10. ——

11. **Solution.**

(a) We have

$$\begin{aligned} 35 &= 2 \cdot 14 + 7 \\ 14 &= 2 \cdot 7 + 0 \end{aligned}$$

$\gcd(35, 14) = 7$.

(b) We have

$$\begin{aligned} 15 &= 1 \cdot 11 + 4 \\ 11 &= 2 \cdot 4 + 3 \\ 4 &= 1 \cdot 3 + 1 \\ 3 &= 3 \cdot 1 + 0 \end{aligned}$$

$\gcd(15, 11) = 1$.

(c) We have

$$\begin{aligned} 252 &= 1 \cdot 180 + 72 \\ 180 &= 2 \cdot 72 + 36 \\ 72 &= 2 \cdot 36 + 0 \end{aligned}$$

$\gcd(252, 180) = 36$.

(d) We have

$$\begin{aligned} 1368 &= 1 \cdot 1278 + 90 \\ 1278 &= 14 \cdot 90 + 18 \\ 90 &= 5 \cdot 18 \end{aligned}$$

Hence $\gcd(1368, 1278) = 18$.

12. **Solution.**

(a) i. Let $n \in \mathbb{N}\backslash\{0,1\}$. Applying the Euclidean Algorithm to the pair of positive integers $n, n+1$, we obtain

$$\begin{array}{rcl} n+1 & = & 1 \cdot n + 1 \\ n & = & n \cdot 1 + 0 \end{array}$$

It follows that $\gcd(n, n+1) = 1$.

ii. Let $n \in \mathbb{N}\backslash\{0,1\}$. Applying Euclidean Algorithm to the pair of positive integers $2n-1, 2n+1$, we obtain

$$\begin{array}{rcl} 2n+1 & = & 1 \cdot (2n-1) + 2 \\ 2n-1 & = & (n-1) \cdot 2 + 1 \\ 2 & = & 2 \cdot 1 + 0 \end{array}$$

It follows that $\gcd(2n-1, 2n+1) = 1$.

(b) The following statement holds:

- Let $n \in \mathbb{N}\backslash\{0,1\}$. If n is an odd integer then $\gcd(3n-1, 3n+1) = 2$. If n is an even integer then $\gcd(3n-1, 3n+1) = 1$.

Justification:

- Let $n \in \mathbb{N}\backslash\{0,1\}$.
  * Suppose n is an odd integer. Then there exists some $k \in \mathbb{Z}$ such that $n = 2k+1$. Therefore $3n-1 = 6k+2$, and $3n+1 = 6k+4$.
  Since $n \geq 2$, we have $k \geq 1$.
  Applying Euclidean Algorithm to the pair of positive integers $6k+2, 6k+4$, we obtain

  $$\begin{array}{rcl} 6k+4 & = & 1 \cdot (6k+2) + 2 \\ 6k+2 & = & (3k+1) \cdot 2 + 0 \end{array}$$

  Then $\gcd(3n-1, 3n+1) = \gcd(6k+2, 6k+4) = 2$.
  * Suppose n is an even integer. Then there exists some $k \in \mathbb{Z}$ such that $n = 2\ell$. Therefore $3n-1 = 6\ell-1$, and $3n+1 = 6\ell+1$.
  Since $n \geq 2$, we have $\ell \geq 1$.
  Applying Euclidean Algorithm to the pair of positive integers $6k+2, 6k+4$, we obtain

  $$\begin{array}{rcl} 6\ell+1 & = & 1 \cdot (6\ell-1) + 2 \\ 6\ell-1 & = & (3\ell-1) \cdot 2 + 1 \\ 2 & = & 2 \cdot 1 + 0 \end{array}$$

  Then $\gcd(3n-1, 3n+1) = \gcd(6\ell-1, 6\ell+1) = 1$.

13. (a) *Hint.* Re-write the equality $\left(\begin{array}{c} p \\ r \end{array}\right) = \dfrac{p!}{(r!) \cdot [(p-r)!]}$ as $(r!) \cdot [(p-r)!] \cdot \left(\begin{smallmatrix} p \\ r \end{smallmatrix}\right) = p \cdot [(p-1)!]$.

What does Euclid's Lemma tells you in light of p being a prime number?

(b) *Hint.* Apply the Binomial Theorem.

(c) *Hint.* Apply mathematical induction to the proposition $S(n)$ stated below:

$n^p \equiv n (\text{mod } p)$.

(d) *Hint.* Apply the result in part (c).

14. (a) **Solution.**

Regard $0, 1, 2$ as distinct objects.
Let $A = \{0,1\}$, $B = \{1\}$, $C = \{2\}$.
We have $A \cap B = B = \{1\}$, $C \backslash B = C = \{2\}$, $A \backslash (C \backslash B) = A = \{0,1\}$.
Note that $0 \in A \backslash (C \backslash B)$ and $0 \notin A \cap B$.
Hence $A \backslash (C \backslash B) \not\subset A \cap B$.

Remark. Be aware that to dis-prove the statement stated in the question is to prove its negation, which reads:

- *There exist some sets A, B, C such that $A \backslash (C \backslash B) \not\subset A \cap B$.*

(b) **Solution.**

Regard $0, 1, 2$ as distinct objects.
Let $A = \{0\}$, $B = \{1\}$, $C = \{2\}$.
A, B, C are non-empty sets.
We have $B \backslash A = B = \{1\}$, $C \backslash A = C = \{2\}$, $C \backslash B = C = \{2\}$, and $(C \backslash A) \backslash (C \backslash B) = \emptyset$.
Note that $1 \in B \backslash A$ and $1 \notin (C \backslash A) \backslash (C \backslash B)$.
Hence $B \backslash A \not\subset (C \backslash A) \backslash (C \backslash B)$.

Remark. Be aware that to dis-prove the statement stated in the question is to prove its negation, which reads:

- *There exist some non-empty sets A, B, C such that $B \backslash A \not\subset (C \backslash A) \backslash (C \backslash B)$.*

(c) **Solution.**
Regard $0, 1, 2$ as distinct objects.
Let $A = \{0\}, B = \{1\}, C = \{2\}$.
We have $B \cap C = \emptyset$. Then $A \cup (B \cap C) = \{0\}$.
We also have $A \cup B = \{0, 1\}$. Then $(A \cup B) \cap C = \emptyset$.
Note that $0 \in A \cup (B \cap C)$ and $0 \neq (A \cup B) \cap C$
Here $A \cup (B \cap C) \not\subset (A \cup B) \cap C$.
Remark. Be aware that to dis-prove the statement stated in the question is to prove its negation, which reads:

- *There exist some non-empty sets A, B, C such that $A \cup (B \cap C) \not\subset (A \cup B) \cap C$.*

(d) ——

(e) ——

(f) ——

(g) ——

15. **Solution.**

(a) Let A, B be sets.

- Suppose $A \subsetneqq B$. Then $A \subset B$ and $A \neq B$.
In particular $A \subset B$.
Since $A \neq B$, we have ($A \not\subset B$ or $B \not\subset A$).
Since $A \subset B$, we have $B \not\subset A$.
- Suppose $A \subset B$ and $B \not\subset A$.
Since $B \not\subset A$, it is not true that $B \subset A$. Therefore $A \neq B$.
It follows that $A \subsetneqq B$.

(b) Let A, B, C be sets. Suppose $A \subset B$ and $B \subset C$. Further suppose $A \subsetneqq B$ or $B \subsetneqq C$.
Since $A \subset B$ and $B \subset C$, we have $A \subset C$.
We verify that $C \not\subset A$:

- Suppose it were true that $C \subset A$. Then, since $A \subset C$, we would have $A = C$. Moreover, since $A \subset B$ and $B \subset C = A$, we would have $A = B$.
Therefore $B = C$ also.
Now $A = B$ and $B = C$. Then ($A \subsetneqq B$ or $B \subsetneqq C$) would be false.
Contradiction arises. Hence $C \not\subset A$ in the first place.

Now we have $A \subset C$ and $C \not\subset A$. It follows that $A \subsetneqq C$.

16. ——

17. **Solution.**
Let M be a set, and C be a subset of $\mathfrak{P}(M)$.
Define $I = \{x \in M : x \in V \text{ for any } V \in C\}$, $J = \{x \in M : x \in V \text{ for some } V \in C\}$.

(a) Let P be a subset of M. Suppose $P \subset V$ for any $V \in C$.
Pick any $x \in P$.

- Pick any $V \in C$. Now we have $x \in P$ and $P \subset V$. Then by assumption, we have $x \in V$.

Therefore $x \in I$ by the definition of I.
It follows that $P \subset I$.

(b) Let Q be a subset of M. Suppose $V \subset Q$ for any $V \in C$.
Pick any $x \in J$. By the definition of J, there exists some $V_x \in C$ such that $x \in V_x$. By assumption, $V_x \subset Q$.
Now we have $x \in V_x$ and $V_x \subset Q$. Therefore $x \in Q$.
It follows that $J \subset Q$.

(c) Let R be a subset of M. Suppose $D = \{V \cap R \mid V \in C\}$, and $K = \{x \in M : x \in U \text{ for some } U \in D\}$.

- Pick any $x \in K$. By the definition of K, there exists some $U_x \in D$ such that $x \in U_x$.
By the definition of D, there exists some $V_x \in C$ such that $U_x = V_x \cap R$.
Then $x \in V_x \cap R$. Therefore $x \in V_x$ and $x \in R$.
Since $x \in V_x$ and $V_x \in C$, we have $x \in J$ by the definition of J.
Now $x \in J$ and $x \in R$. Then $x \in J \cap R$.

- Pick any $x \in J \cap R$. We have $x \in J$ and $x \in R$.
  In particular, $x \in J$. Then there exists some $V_x \in C$ such that $x \in V_x$.
  Now $x \in V_x$ and $x \in R$. Then $x \in V_x \cap R$.
  Define $U_x = V_x \cap R$. By the definition of U_x, we have $U_x \in D$, and $x \in U_x$.
  Then by the definition of K, we have $x \in K$.

It follows that $K = J \cap R$.