1. (a) Let P, Q be statements. Verify that the statement $[(P \to Q) \wedge P] \to Q$ is a tautology by drawing an appropriate truth table.
   (b) Let P, Q be statements. Consider the statement $[(P \to Q) \wedge Q] \to P$. Determine whether it is a tautology, or a contradiction, or a contingent statement. Justify your answer by drawing an appropriate truth table.

2. Let P, Q be statements. By drawing one or more appropriate truth tables, verify that the statements

$$P \leftrightarrow (\sim Q), \quad (\sim P) \leftrightarrow Q, \quad (P \vee Q) \wedge [\sim(P \wedge Q)], \quad [P \wedge (\sim Q)] \vee [(\sim P) \wedge Q]$$

are logically equivalent to each other.

Remark. $(P \vee Q) \wedge [\sim(P \wedge Q)]$ is called the **exclusive disjunction** of P, Q. We may denote it by $P \underline{\vee} Q$. It is true exactly when one and only one of P, Q is true. In words we write 'P xor Q', or as 'either P or Q'. It is the kind of 'or' that you find in 'coffee or tea' in a restaurant menu.

3. Let P, Q, R be statements. Consider each of the pairs of statements below. Determine whether the statements are logically equivalent. Justify your answer by drawing an appropriate truth table.
   (a) $P \to (Q \wedge R)$, $(P \to Q) \wedge (P \to R)$.
   (b) $P \to (Q \to R)$, $(P \wedge Q) \to R$.
   (c) $P \to (Q \vee R)$, $(P \to Q) \vee (P \to R)$.
   (d) $(P \vee Q) \to R$, $(P \to R) \wedge (Q \to R)$.
   (e) $(P \to Q) \to R$, $P \to (Q \to R)$.
   (f) $P \to (Q \vee R)$, $[P \wedge (\sim Q)] \to R$.

4. Let P, Q, R be statements. Consider each of the statements below. Determine whether it is a tautology or a contradiction or a contingent statement . Justify your answer by drawing an appropriate truth table.
   (a) $[P \to (P \to Q)] \to (P \to Q)$
   (b) $(P \to R) \to [(P \wedge Q) \to R)]$
   (c) $[(P \to Q) \wedge (Q \to R)] \to (P \to R)$
   (d) $[(P \to Q) \wedge (Q \to R) \wedge (R \to P)] \to (Q \to P)$
   (e) $(P \to R) \to [(P \to Q) \vee (Q \to R)]$
   (f) $(P \to Q) \to [(Q \to R) \vee (P \wedge R)]$

5. *You are not required to justify your answer.*
   Consider each of the sets below. List every element of the set concerned, each exactly once.
   (a) $A = \{0, 1, 2, 3\}$.
   (b) $B = \{0, 0, 1, 2, 3, 1, 4\}$.
   (c) $C = \{0, 1, 2, \{0\}, \{1\}\}$.
   (d) $D = \{0, 1, \{0, 1\}\}$.
   (e) $E = \{0, 1, \{0, 1\}, \{\{0, 1\}\}\}$.

6. *You are not required to justify your answer.*
   Let $A = \{0, 1, 2, 3\}$, $B = \{0, 0, 2, 1, 0\}$, $C = \{1, 3, 3, 1, 0, 3\}$, $D = \{0, 2, \{1\}\}$, $E = \{0, \{1\}, \{2, 3\}\}$, $F = \{0, \{2\}, \{3\}\}$, $G = \{0, \{1, 2\}, \{3\}\}$.
   Consider each of the sets below. List every element of the set concerned, each element exactly once. Where the set concerned is the empty set, write '*this set is the empty set*'.

(a) A	(d) $B \cap D$	(g) $A \triangle G$	(j) $F \backslash E$
(b) B	(e) $A \cup E$	(h) $D \backslash E$	(k) $G \backslash F$
(c) C	(f) $A \cup G$	(i) $E \backslash D$	(l) $\mathfrak{P}(D)$

7. *You are not required to justify your answer.*
   Let $C = \{0, 1, 1, 2, 3, 3, 4\}$, $D = \{0, 1, \{1, 2, 3\}, \{\{3\}, 4\}\}$.
   Consider each of the sets below. List every element of the set concerned, each element exactly once. Where the set concerned is the empty set, write '*this set is the empty set*'.

(a)	C.	(c)	$C \cap D$.	(e)	$C \backslash D$.	(g)	$C \triangle D$.
(b)	D.	(d)	$C \cup D$.	(f)	$D \backslash C$.	(h)	$\mathfrak{P}(C \cap D)$.

8. *You are not required to justify your answer.*

   Let $A = \{\{3,5,3\},5,7,7\}$, $B = \{\{3,5\},\{5,7\}\}$, $C = \{1,\{5\}\}$.

   Consider each of the sets below. List every element of the set concerned, each element exactly once. Where the set concerned is the empty set, write '*this set is the empty set*'.

   (a) $A \cap B$ (b) $B \cup C$ (c) $B \backslash A$ (d) $\mathfrak{P}(B)$

9. *You are not required to justify your answer.*

   Let $B = \{\{b,e\},\{e\},\{t\},\{h\},\{o,v\},\{e\},\{n\}\}$, $H = \{\{h\},\{a,y,d\},\{n\}\}$, $M = \{\{m,o\},\{z,a,r,t\}\}$, $S = \{\{s,c\},\{h\},\{u\},\{b,e\},\{r\},\{t\}\}$.

   Consider each of the sets below. List every element of the set concerned, each element exactly once. Where the set concerned is the empty set, write '*this set is the empty set*'.

   (a) $B \cap H$ (b) $B \cup H$ (c) $B \backslash S$ (d) $\mathfrak{P}(M)$

10. *You are not required to justify your answers in this question.*

    Let $C = \{c,a,n,t,o,r\}$, $D = \{d,e,d,e,k,i,n,d\}$, $K = \{k,r,o,n,e,c,k,e,r\}$.

    (a) How many elements are there in the set $C \cup D$?
    (b) How many elements are there in the set $\{C\} \cup \{D\}$?
    (c) How many elements are there in the set $\{C \cup D\}$?
    (d) List every element of the set $C \backslash K$, each element exactly once.
    (e) List every element of the set $\mathfrak{P}(C \backslash K)$, each element exactly once.

11. *You are not required to justify your answers in this question.*

    Let $S = \{s,e,n,a,t,u,s\}$, $P = \{p,o,p,u,l,u,s,q,u,e\}$, $R = \{r,o,m,a,n,u,s\}$.

    (a) How many elements are there in the set P?
    (b) How many elements are there in the set $S \cup R$?
    (c) How many elements are there in the set $(S \cup R) \cap P$?
    (d) How many elements are there in the set $S \cup (R \cap P)$?
    (e) How many elements are there in the set $\{S\} \cup \{R \cap P\}$?
    (f) How many elements are there in the set $\{S \cup P\} \backslash \{R\}$?
    (g) List every element of the set $S \cap P \cap R$, each element exactly once.
    (h) List every element of the set $\mathfrak{P}(S \cap P \cap R)$, each element exactly once.

12. *You are not required to justify your answers in this question.*

    Let $Q = \{q,u,o,d\}$, $E = \{e,r,a,t\}$, $D = \{d,e,m,o,n,s,t,r,a,n,d,u,m\}$.

    (a) How many elements are there in the set D?
    (b) How many elements are there in the set $(Q \cup E) \backslash D$?
    (c) How many elements are there in the set $Q \cup (E \backslash D)$?
    (d) How many elements are there in the set $\{E\} \cup \{D \cap E\}$?
    (e) List every element of the set $Q \cap D$, each element exactly once.
    (f) List every element of the set $\mathfrak{P}(Q \cap D)$, each element exactly once.

13. *You are not required to justify your answers in this question.*

    Let $E = \{\emptyset, \mathbb{N}\}$, $F = \{\emptyset, \{\mathbb{N}\}\}$, $G = \{\{\emptyset\}, \mathbb{N}\}$, $H = \{\{\emptyset\}, \{\mathbb{N}\}\}$.

    (a) How many elements are there in the set E?
    (b) How many elements are there in the set $E \cup F \cup G \cup H$?
    (c) How many elements are there in the set $\{E, F, H\}$?
    (d) List every element of the set $E \cap F$, each element exactly once.
    (e) List every element of the set $G \backslash H$, each element exactly once.
    (f) List every element of the set $\mathfrak{P}(G) \backslash (E \cup G)$, each element exactly once.

14. *You are not required to justify your answers in this question.*

Let $A = \{x \in \mathbb{N}\backslash\{0,1\} : x^2 = n^3 \text{ for any } n \in \mathbb{Z}\}$, $B = \{x \in \mathbb{N}\backslash\{0,1\} : x^2 = n^3 \text{ for some } n \in \mathbb{Z}\}$.

(a) Is A the empty set? If *yes*, just write '$A = \emptyset$'. If *no*, write '$A \neq \emptyset$' and name one element of A.

(b) Is B the empty set? If *yes*, just write '$B = \emptyset$'. If *no*, write '$B \neq \emptyset$' and name one element of B.

15. *You are not required to justify your answers in this question.*

Let $A = \{x \in \mathbb{N}\backslash\{0\} : x = r^2 - r - 12 \text{ for any } r \in \mathbb{Z}\}$, $B = \{x \in \mathbb{N}\backslash\{0\} : x = r^2 - r - 12 \text{ for some } r \in \mathbb{Z}\}$.

(a) Is A the empty set? If *yes*, just write '$A = \emptyset$'. If *no*, write '$A \neq \emptyset$' and also name one element of A.

(b) Is B the empty set? If *yes*, just write '$B = \emptyset$'. If *no*, write '$B \neq \emptyset$' and also name one element of B.

16. Consider each of the statements below. Determine whether it is true or not. Justify your answer.

You may take for granted that $\sqrt{5}$ is an irrational number whose value is between 2 and 3.

(a) $\sqrt{5} \in \{x \in \mathbb{R} : 1 \leq x < 3\}$.

(b) $\sqrt{5} \in \{x \in \mathbb{R} : 1 \leq x < 2\}$.

(c) $\sqrt{5} \in \{x \in \mathbb{Q} : 1 \leq x < 3\}$.

(d) $\sqrt{5} \in \{x \in \mathbb{R} : x^2 \in \mathbb{N}\}$.

(e) $\sqrt{5} \in \{x \in \mathbb{R} : x = -r^2 \text{ for some } r \in \mathbb{R}\}$.

(f) $\sqrt{5} \in \{x \in \mathbb{R} : x = a + b\sqrt{5} \text{ for some } a, b \in \mathbb{Z}\}$.

17. Consider each of the 'infinite' collections of objects below. Apply the Method of Specification to express the collection as a set.

(a) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots, \frac{1}{2^n}, \frac{1}{2^{n+1}}, \cdots.$

(b) $1, \frac{3}{5}, \frac{9}{25}, \frac{27}{125}, \frac{81}{625}, \cdots, \left(\frac{3}{5}\right)^n, \left(\frac{3}{5}\right)^{n+1}, \cdots.$

(c)

$$\begin{array}{cccccccccc}
1 & \frac{1}{3} & \frac{1}{9} & \frac{1}{27} & \frac{1}{81} & \cdots & \frac{1}{3^n} & \frac{1}{3^{n+1}} & \cdots \\
2 & \frac{2}{3} & \frac{2}{9} & \frac{2}{27} & \frac{2}{81} & \cdots & \frac{2}{3^n} & \frac{2}{3^{n+1}} & \cdots \\
4 & \frac{4}{3} & \frac{4}{9} & \frac{4}{27} & \frac{4}{81} & \cdots & \frac{4}{3^n} & \frac{4}{3^{n+1}} & \cdots \\
8 & \frac{8}{3} & \frac{8}{9} & \frac{8}{27} & \frac{8}{81} & \cdots & \frac{8}{3^n} & \frac{8}{3^{n+1}} & \cdots \\
16 & \frac{16}{3} & \frac{16}{9} & \frac{16}{27} & \frac{16}{81} & \cdots & \frac{16}{3^n} & \frac{16}{3^{n+1}} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \\
2^m & \frac{2^m}{3} & \frac{2^m}{9} & \frac{2^m}{27} & \frac{2^m}{81} & \cdots & \frac{2^m}{3^n} & \frac{2^m}{3^{n+1}} & \cdots \\
2^{m+1} & \frac{2^{m+1}}{3} & \frac{2^{m+1}}{9} & \frac{2^{m+1}}{27} & \frac{2^{m+1}}{81} & \cdots & \frac{2^{m+1}}{3^n} & \frac{2^{m+1}}{3^{n+1}} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots &
\end{array}$$

(d)

$$\begin{array}{cccccccc}
0 & 1-e & 1-e^2 & \cdots & 1-e^n & 1-e^{n+1} & \cdots \\
\pi-1 & \pi-e & \pi-e^2 & \cdots & \pi-e^n & \pi-e^{n+1} & \cdots \\
\pi^2-1 & \pi^2-e & \pi^2-e^2 & \cdots & \pi^2-e^n & \pi^2-e^{n+1} & \cdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \\
\pi^m-1 & \pi^m-e & \pi^m-e^2 & \cdots & \pi^m-e^n & \pi^m-e^{n+1} & \cdots \\
\pi^{m+1}-1 & \pi^{m+1}-e & \pi^{m+1}-e^2 & \cdots & \pi^{m+1}-e^n & \pi^{m+1}-e^{n+1} & \cdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots &
\end{array}$$

18. For each equation with unknown in the reals below, determine its solution set by solving for its general solution.

You are not required to give the 'checking step' explicitly, but be careful not to wrongly include false candidates amongst the solution, nor wrongly ignore a genuine solution.

(a) $\sin(2x) + \cos(x) = 0$.

(b) $\tan^2(x) + 3 = 2\sec^2(x)$.

(c) $\cos(3x) = \cos(x)$.

(d) $\sin(x) + \sin(2x) + \sin(3x) = 0$.

(e) $\sin(2x) + \sin(4x) = \cos(x)$.

(f) $\cos(4x) + \cos(2x) = \cos(x)$.

(g) $2\cos(2x) + 5\sin(x) - 3 = 0$.

(h) $\sin(5x) + \sin(3x) = \cos(x)$.

(i) $12\cos(3x) - 5\sin(3x) = 13$.

(j) $\sin(3x + \frac{\pi}{4})\cos(3x - \frac{\pi}{4}) = \frac{3}{4}$.

(k)$^\diamondsuit$ $\cos(4x) - 2\sin^2(x) = -2\sin^2(\frac{x}{2})$.

(l)$^\diamondsuit$ $\sin(\frac{2}{x}) = \frac{1}{2}$.

(m)$^\diamondsuit$ $\cot(\frac{x^3}{3}) = -\sqrt{3}$.

(n)$^\diamondsuit$ $\cos(4x^2) = -1$.

(o)$^\clubsuit$ $\sin(\frac{1}{\sqrt{x}})\sin(x^2) = 0$.

(p)$^\clubsuit$ $\sin(\frac{2}{x}) = \sin(\frac{1}{x})$.

19.$^\diamondsuit$ Solve for all real solutions of the equation $\sin^2(3\theta) - \sin^2(2\theta) - \sin(\theta) = 0$.

Remark. Express $\sin^2(\mu) - \sin^2(\nu)$ in terms of $\cos(\mu+\nu), \cos(\mu-\nu), \sin(\mu+\nu), \sin(\mu-\nu)$.

20.$^\diamondsuit$ Solve for all real solutions of the equation $\cos^2(2\theta) - \sin^2(3\theta) + \cos(\theta)\sin(5\theta) = 0$.

Remark. Express $\cos^2(\mu) - \sin^2(\nu)$ in terms of $\cos(\mu+\nu), \cos(\mu-\nu), \sin(\mu+\nu), \sin(\mu-\nu)$.

21.$^\diamondsuit$ Solve for all real solutions of the equation $\sin(4x) - \sin(3x) + \sin(2x) - \sin(x) = 0$.

Remark. Express $\sin(4\theta) - \sin(3\theta) + \sin(2\theta) - \sin(\theta)$ in the form $A\sin(\frac{\theta}{2})\cos(p\theta)\cos(q\theta)$. Here A, p, q are some real numbers whose values you have to determine.

22.$^\diamondsuit$ Solve for all real solutions of the equation $(4\cos^2(x) - 3)\sin(2x) = \sin(x)$.

Remark. Express $\cos(3\theta)$ in terms of $\cos(\theta)$.

23. (a) Let $\alpha, \beta, k \in \mathbb{R}$. Suppose $\tan(\alpha), \tan(\beta)$ are well-defined as real numbers. Further suppose that $\tan(\alpha) = k\tan(\beta)$.

i. Prove that $\sin(\alpha + \beta) = (k+1)\cos(\alpha)\sin(\beta)$.

ii. Hence deduce that $(k+1)\sin(\alpha - \beta) = (k-1)\sin(\alpha + \beta)$.

(b) Let $\theta, k \in \mathbb{R}$. Suppose $\tan(\theta + \frac{\pi}{18}), \tan(\theta - \frac{\pi}{9})$ are well-defined as real numbers and $\tan(\theta + \frac{\pi}{18}) = k\tan(\theta - \frac{\pi}{9})$.

i. Prove that $\sin(2\theta - \frac{\pi}{18}) = \frac{k+1}{2(k-1)}$.

ii. Determine all possible values of k.

(c) Solve for all real solutions of the equation $\tan(x + \frac{\pi}{18}) = -2\tan(x - \frac{\pi}{9})$.

24. Let $z = \dfrac{1+\sqrt{3}i}{\sqrt{3}+i}$. Express z in polar form. Hence find $\arg(z^{2018})$.

25. Let $\zeta = 1 + i$. Find:

(a) the square roots of ζ,

(b) the cube roots of ζ,

(c) the quartic roots of ζ.

Leave your answer in polar form.

26. Let $\zeta = \sin(\frac{2\pi}{3}) + i\cos(\frac{2\pi}{3})$. Express ζ in polar form. Hence find the three cube roots of ζ, expressing your answer in polar form.

27. (a) Solve for all real solutions of the equation $\cos(x) = \dfrac{1}{\sqrt{2}}$ with unknown x.

(b) Let m be a positive integer.

i. Let $r, \theta \in \mathbb{R}$, and $z = r(\cos(\theta) + i\sin(\theta))$. Show that $z^m + \bar{z}^m = 2r^m\cos(m\theta)$.

ii. Express $\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^m+\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^m$ in the form $A\cos(B\theta)$, in which A, B are real numbers, possibly dependent on m.

iii. Hence or otherwise, find all possible values of m for which $\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^m+\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^m=\sqrt{2}$.

28. (a) Let p be a positive integer. Find all possible values of p for which $(1+i)^p-(1-i)^p=0$.

(b) Hence, or otherwise, find the value of $\frac{(1+i)^{4k+1}}{(1-i)^{4k-1}}$, where k is a positive integer.

29. (a) Let θ be a real number. Suppose θ is not an integral multiple of π.

Let $\omega=\cos(\theta)+i\sin(\theta)$. Suppose $\omega^2-2\bar{\omega}+\frac{1}{\omega}$ is real. Find the two possible values of ω.

(b) Denote by α, β the two values of ω obtained in the previous part.

i. Prove that $\alpha^2=\beta$ and $\beta^2=\alpha$.

ii. Find the respective values of α^3 and β^3.

iii. Suppose n is an integer. Find all possible values of $\alpha^n+\beta^n$.

30. (a) i. Let $\theta\in\mathbb{R}$, and $\omega=\cos(\theta)+i\sin(\theta)$. Let n be a positive integer.

Prove that $\omega^n+\frac{1}{\omega^n}=2\cos(n\theta)$ and $\omega^n-\frac{1}{\omega^n}=2i\sin(n\theta)$.

ii. Hence find all complex numbers which satisfies $\frac{(z^2-1/z^2)i}{z^2+1/z^2}=-\sqrt{3}$.

(b) Consider the polynomial $f(x)=x^2-x+1$.

i. Express, in polar form, the roots of $f(x)$.

ii. Let α, β be the roots of $f(x)$. Let k be an integer. Find the quadratic polynomial with leading coefficient 1 whose roots are $\left(\frac{\alpha}{\beta}\right)^k$ and $\left(\frac{\beta}{\alpha}\right)^k$ for the various scenarios below:

A. $k=3p$ for some $p\in\mathbb{Z}$.

B. $k=3q+1$ for some $q\in\mathbb{Z}$.

C. $k=3r+2$ for some $r\in\mathbb{Z}$.

31.$^\diamondsuit$ Let $\theta\in\mathbb{R}$ and $z=\cos(\theta)+i\sin(\theta)$.

(a) Let $n\in\mathbb{N}\backslash\{0\}$. Prove that $2^n\cos^n(\theta)=\left(z+\frac{1}{z}\right)^n$ and $2^n i^n\sin^n(\theta)=\left(z-\frac{1}{z}\right)^n$.

(b) Hence, or otherwise, deduce the results below:

i. $\cos^7(\theta)=\frac{1}{2^C}\sum_{j=0}^{7}A_j\cos(j\theta)$ *and* $\sin^7(\theta)=\frac{1}{2^D}\sum_{j=0}^{7}B_j\sin(j\theta)$.

Here $A_0, A_1, A_2, ..., A_7$, $B_0, B_1, B_2, ..., B_7$, C, D *are integers whose respective value you have to determine.*

ii. $\cos^8(\theta)=\frac{1}{2^C}\sum_{j=0}^{8}A_j\cos(j\theta)$ *and* $\sin^8(\theta)=\frac{1}{2^D}\sum_{j=0}^{8}B_j\cos(j\theta)$.

Here $A_0, A_1, A_2, ..., A_8$, $B_0, B_1, B_2, ..., B_8$, C, D *are integers whose respective value you have to determine.*

Remark. Can you generalize the idea for the situation of $\cos^n(\theta)$, $\sin^n(\theta)$, in which n is an arbitrary positive integer?

32.$^\heartsuit$ Let $n\in\mathbb{N}$. Let α, θ be real numbers. Suppose $\sin(\theta/2)\neq 0$.

(a) Prove that

$$\sum_{k=0}^{n}\cos(\alpha+k\theta)=\frac{\sin((n+1)\theta/2)\cos(\alpha+n\theta/2)}{\sin(\theta/2)}, \text{ and } \sum_{k=0}^{n}\sin(\alpha+k\theta)=\frac{\sin((n+1)\theta/2)\sin(\alpha+n\theta/2)}{\sin(\theta/2)}.$$

(b) Suppose $n\geq 3$. Write $\varphi=\frac{2\pi}{n}$. Let $\omega_n=\cos(\varphi)+i\sin(\varphi)$. Let $\eta\in\mathbb{C}$. Suppose $|\eta|=1$.

Prove that $\sum_{k=0}^{n-1}|\eta-{\omega_n}^k|^2=2n$.

Remark. Below is the geometric interpretation of the result. Consider the regular n-sided polygon with vertices $1, \omega_n, {\omega_n}^2, \cdots, {\omega_n}^{n-1}$. The sum of the squares of the lengths of the n chords joining respectively the n points of this polygon with any one point on the unit circle with centre 0 is the constant n.

33. Write $\theta = \frac{2\pi}{5}$, $\omega = \cos(\theta) + i\sin(\theta)$, $\sigma = \omega + \omega^4$, $\tau = \omega^2 + \omega^3$. Let $g(u)$ be the polynomial given by $g(u) = u^2 + u - 1$.

 (a) Verify that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$.

 (b) Verify that σ, τ are the distinct real roots of the polynomial $g(u)$.

 (c) i. Prove that $\mathsf{Re}(\omega) = a + b\sqrt{M}$. Hence, or otherwise, find the value of ω.
 Here a, b are rational numbers and M is an integer. You have to determine the explicit value of a, b, M.

 ii. Prove that $\mathsf{Re}(\omega^2) = c + d\sqrt{N}$. Hence, or otherwise, find the value of ω^2.
 Here c, d are rational numbers and N is an integer. You have to determine the explicit value of c, d, N.

 iii. Find the respective values of ω^3, ω^4.

34. Let $A = \{x \in \mathbb{R} : x^2 - 2x - 3 \leq 0\}$, $B = \{x \in \mathbb{R} : -1 \leq x \leq 3\}$.
Prove that $A = B$.

35. Let $A = \{x \in \mathbb{Z} : x = k^4 \text{ for some } k \in \mathbb{Z}\}$, $B = \{x \in \mathbb{Z} : x = k^2 \text{ for some } k \in \mathbb{Z}\}$.

 (a) Prove that $A \subset B$.

 (b) Prove that $B \not\subset A$.

36. Let $A = \{x \mid x = 54m^6 \text{ for some } m \in \mathbb{Z}\}$, $B = \{x \mid x = 2m^3 \text{ for some } m \in \mathbb{Z}\}$.

 (a) Prove that $A \subset B$.

 (b) Prove that $B \not\subset A$.

37. *In this question you may take for granted the validity of the statements below:*

 - *Suppose p, q be distinct positive prime numbers. Then $\sqrt{pq}$ is irrational.*
 - *$\sqrt{6}$ is an irrational number.*

 Let $A = \{x \mid x = s + t\sqrt{2} \text{ for some } s, t \in \mathbb{Z}\}$, $B = \{x \mid x = u + v\sqrt{3} \text{ for some } u, v \in \mathbb{Z}\}$.

 (a)$^\diamondsuit$ Prove that $A \not\subset B$.

 (b)$^\clubsuit$ Prove that $A \cap B = \mathbb{Z}$.

38.$^\diamondsuit$ Let $A = \{n \in \mathbb{Z} : n \equiv 1 (\mathsf{mod}\ 3)\}$, $B = \{n \in \mathbb{Z} : n \equiv 4 (\mathsf{mod}\ 9)\}$.

 (a) Prove that $B \subset A$.

 (b) Prove that $A \not\subset B$.

39. Let $C = \{\zeta \in \mathbb{C} : \mathsf{Re}(\zeta) \geq 0\}, D = \{\zeta \in \mathbb{C} : \mathsf{Im}(\zeta) \geq 0\}, E = \{\zeta \in \mathbb{C} : |\zeta - 1 - i| \leq 1\}$.

 (a)$^\diamondsuit$ Prove that $E \subset C \cap D$.

 (b) Prove that $C \not\subset D$.

 (c) Prove that $D \not\subset C$.

40. Let $A = \{z \mid z = 8^m(\cos(6m) + i\sin(6m)) \text{ for some } m \in \mathbb{Z}\}$, $B = \{z \mid z = 2^m(\cos(2m) + i\sin(2m)) \text{ for some } m \in \mathbb{Z}\}$.

 (a) Prove that $A \subset B$.

 (b) Prove that $B \not\subset A$.

41. Let $A = \{z \in \mathbb{C} : z^2 = ri \text{ for some } r \in \mathbb{R}\}$, $B = \{z \in \mathbb{C} : z^6 = ri \text{ for some } r \in \mathbb{R}\}$.

 (a) Prove that $A \subset B$.

 (b)$^\diamondsuit$ Prove that $B \not\subset A$.

 Remark. The *polar form for complex numbers* and *De Moivre's Theorem* are useful.

42.$^\diamondsuit$ Let $A = \{x \in \mathbb{R} : x^2 - x \geq 0\}$, $B = \{x \in \mathbb{R} : x \leq 0\}$, $C = \{x \in \mathbb{R} : x \geq 1\}$.
Prove that $A = B \cup C$.

43.$^\diamondsuit$ Let $A = \{x \in \mathbb{Q} : x = r^3 \text{ for some } r \in \mathbb{Q}\}$, $B = \{x \in \mathbb{Q} : x = r^9 \text{ for some } r \in \mathbb{Q}\}$,
$C = \{x \in \mathbb{Z} : x = r^3 \text{ for some } r \in \mathbb{Q}\}$, $D = \{x \in \mathbb{Q} : x = r^3 \text{ for some } r \in \mathbb{Z}\}$.

 (a) Is A a subset of $\mathbb{Q}$? Is $\mathbb{Q}$ a subset of A? Justify your answer.

 (b) Is A a subset of B? Is B a subset of A? Justify your answer.

 (c) Is A a subset of C? Is C a subset of A? Justify your answer.

(d) Is A a subset of D? Is D a subset of A? Justify your answer.

44.$^\diamondsuit$ Let $A = \{\zeta \mid \zeta = m^3 + n^4 i \text{ for some } m, n \in \mathbb{N}\}$, $B = \{\zeta \mid \zeta = m + n^8 i \text{ for some } m, n \in \mathbb{N}\}$.

(a) Is it true that $A \subset B$? Justify your answer.

(b) Is it true that $B \subset A$? Justify your answer.

45. Prove each of the statements below 'from first principles', using the definitions of set equality, intersection, union, complement, where appropriate.

(a) *Suppose A, B are sets. Then $A \cap B \subset A$.*

(b) *Suppose A, B be sets. Then $A \subset A \cup B$.*

(c) *Suppose A, B, C be sets. Then $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.*

(d) *Suppose A, B, C be sets. Then $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$.*

(e) *Suppose A, B, C be sets. Then $(A \cap B) \backslash C = (A \backslash C) \cap (B \backslash C)$.*

(f) *Suppose A, B, C be sets. Then $(A \cup B) \backslash C = (A \backslash C) \cup (B \backslash C)$.*

(g) *Suppose A, B are sets. Then $(A \cup B) \backslash A = B \backslash (A \cap B)$.*

(h) *Suppose A, B, C are sets. Then $A \backslash (B \backslash C) = (A \backslash B) \cup (A \cap C)$.*

46. Prove the statements below. *Where appropriate, you may apply results concerned with the properties of the set operations 'intersection', 'union' and 'complement'.*

(a)$^\diamondsuit$ *Suppose A, B are sets. Then $A \triangle B = (A \cup B) \backslash (A \cap B)$.*

(b) *Suppose A, B are sets. Then $A \triangle B = B \triangle A$.*

(c)$^\heartsuit$ *Suppose A, B, C are sets. Then $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.*

(d)$^\clubsuit$ *Suppose A, B, C be sets. Then $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$.*

47.$^\diamondsuit$ Let A, B be sets. Prove that the statements below are logically equivalent 'from first principles', using the definitions of set equality, subset relation, intersection, union, complement, where appropriate:

(I) $A \subset B$. (II) $A \cap B = A$. (III) $A \cup B = B$.

48.$^\heartsuit$ Prove that the Well-ordering Principle for integers, the (set-theoretic formulation of the) Principle of Mathematical Induction, the (set-theoretic formulation of the) Second Principle of Mathematical Induction are logically equivalent:

- **Well-ordering Principle for integers**.
  Suppose S be a non-empty subset of $\mathbb{N}$. Then S has a least element.
- **Principle of Mathematical Induction**.
  Let T be a subset of $\mathbb{N}$. Suppose $0 \in T$. Further suppose that for any $k \in \mathbb{N}$, if $k \in T$ then $k+1 \in T$. Then $T = \mathbb{N}$.
- **Second Principle of Mathematical Induction**.
  Let U be a subset of $\mathbb{N}$. Suppose $0 \in U$. Further suppose that for any $k \in \mathbb{N}$, if $[\![0, k]\!] \subset U$ then $k+1 \in U$. Then $U = \mathbb{N}$.