MATH1050 Exercise 6 (Answers and selected solution)

1. **Answer.**

(a) The statement $(P \to R) \to [(P \to Q) \wedge (Q \to R)]$ is neither a tautology nor a contradiction; it is a contingent statement.

P	Q	R	$P \to Q$	$Q \to R$	$P \to R$	$(P \to Q) \wedge (Q \to R)$	$(P \to R) \to [(P \to Q) \wedge (Q \to R)]$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	F
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	F
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Remark. Do you see what is wrong in the logic behind the word 'therefore' in the argument below?

- If John visits his girlfriend then John puts aside his books. Therefore, if John visits his girlfriend then John plays football; furthermore, if John plays football then John puts aside his books.

(b) The statement $(P \to Q) \to [(P \to R) \vee (Q \to R)]$ is neither a tautology nor a contradiction; it is a contingent statement.

P	Q	R	$P \to Q$	$P \to R$	$Q \to R$	$(P \to R) \vee (Q \to R)$	$(P \to Q) \to [(P \to R) \vee (Q \to R)]$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	F	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

2. **Answer.**

(a) $\{0,1\}$, $\{1,2,3\}$.

(b) $\{0,1\}$, $\{1\}$, $\{1,2,3\}$, $\{3,4\}$, $\{\{3\},\{4\}\}$.

(c) $\{1\}$, $\{3,4\}$.

(d) $\{1\}$,$\{3,4\}$, $\{\{3\},\{4\}\}$.

(e) $\emptyset$, $\{\{1\}\}$, $\{\{3,4\}\}$, $\{\{1\},\{3,4\}\}$.

3. **Answer.**

(a) Five.

(b) Nine.

(c) Six.

(d) One.

(e) Two.

(f) One.

(g) c, r

(h) $\emptyset, \{c\}, \{r\}, \{c,r\}$.

4. **Answer.**

(a) $A = \emptyset$.

(b) $B \neq \emptyset$; 125 is an element of B.

5. (a) **Solution.**

We proceed to solve the equation (†):

$$\begin{aligned} \cos(2x) &= \sin(x) \quad \text{——} \quad (\dagger) \\ 1 - 2\sin^2(x) &= \sin(x) \\ 2\sin^2(x) + \sin(x) - 1 &= 0 \\ (\sin(x)+1)(2\sin(x)-1) &= 0 \end{aligned}$$

$$\sin(x) = -1 \qquad \text{or} \qquad \sin(x) = \frac{1}{2}$$

(Case 1).

$$\begin{aligned} \sin(x) &= -1 \\ x &= -\frac{\pi}{2} + K \cdot 2\pi \quad \text{where } K \in \mathbb{Z} \end{aligned}$$

(Case 2).

$$\begin{aligned} \sin(x) &= \frac{1}{2} \\ x &= (-1)^M \cdot \frac{\pi}{6} + M\pi \quad \text{where } M \in \mathbb{Z} \end{aligned}$$

The general solution of (†) is given by $x = -\frac{\pi}{2} + K \cdot 2\pi$ where $K \in \mathbb{Z}$, or $x = (-1)^M \cdot \frac{\pi}{6} + M\pi$ where $M \in \mathbb{Z}$. Define

$$\begin{aligned} A &= \left\{ x \in \mathbb{R} : x = -\frac{\pi}{2} + K \cdot 2\pi \text{ for some } K \in \mathbb{Z} \right\}, \\ B &= \left\{ x \in \mathbb{R} : x = (-1)^M \cdot \frac{\pi}{6} + M\pi \text{ for some } M \in \mathbb{Z} \right\}. \end{aligned}$$

The solution set of (†) is given by $A \cup B$.

(b) **Solution.**

We proceed to solve the equation (†):

$$\begin{aligned} \sin(2x) + \sin(8x) &= \sin(5x) \quad \text{——} \quad (\dagger) \\ \sin(5x - 3x) + \sin(5x + 3x) &= \sin(5x) \\ 2\sin(5x)\cos(3x) &= \sin(5x) \\ (2\cos(3x) - 1)\sin(5x) &= 0 \\ \cos(3x) = \frac{1}{2} \quad &\text{or} \quad \sin(5x) = 0 \end{aligned}$$

(Case 1).

$$\begin{aligned} \cos(3x) &= \frac{1}{2} \\ 3x &= \pm\frac{\pi}{3} + K \cdot 2\pi \quad \text{where } K \in \mathbb{Z} \\ x &= \pm\frac{\pi}{9} + K \cdot \frac{2\pi}{3} \end{aligned}$$

(Case 2).

$$\begin{aligned} \sin(5x) &= 0 \\ 5x &= M\pi \quad \text{where } M \in \mathbb{Z} \\ x &= M \cdot \frac{\pi}{5} \end{aligned}$$

The general solution of (†) is given by $x = \pm\frac{\pi}{9} + K \cdot \frac{2\pi}{3}$ where $K \in \mathbb{Z}$, or $x = M \cdot \frac{\pi}{5}$ where $M \in \mathbb{Z}$. Define

$$\begin{aligned} A &= \left\{ x \in \mathbb{R} : x = \frac{\pi}{9} + K \cdot \frac{2\pi}{3} \text{ for some } K \in \mathbb{Z} \right\}, \\ B &= \left\{ x \in \mathbb{R} : x = -\frac{\pi}{9} + L \cdot \frac{2\pi}{3} \text{ for some } L \in \mathbb{Z} \right\}, \\ C &= \left\{ x \in \mathbb{R} : x = M \cdot \frac{\pi}{5} \text{ for some } M \in \mathbb{Z} \right\} \end{aligned}$$

The solution set of (†) is given by $A \cup B \cup C$.

(c) **Solution.**

We proceed to solve the equation (†):

$$\begin{aligned} 2\sin(\frac{x}{2})\sin(\frac{3x}{2}) &= 1 \quad \text{——} \quad (\dagger) \\ \cos(\frac{x}{2}-\frac{3x}{2})-\cos(\frac{x}{2}+\frac{3x}{2}) &= 1 \\ \cos(-x)-\cos(2x) &= 1 \\ \cos(x)-(2\cos^2(x)-1) &= 1 \\ (1-2\cos(x))\cos(x) &= 0 \\ \cos(x)=\frac{1}{2} \quad &\text{or} \quad \cos(x)=0 \end{aligned}$$

(Case 1).

$$\begin{aligned} \cos(x) &= \frac{1}{2} \\ x &= \pm\frac{\pi}{3}+K\cdot 2\pi \quad \text{where } K\in\mathbb{Z} \end{aligned}$$

(Case 2).

$$\begin{aligned} \cos(x) &= 0 \\ x &= \frac{\pi}{2}+M\pi \quad \text{where } M\in\mathbb{Z} \end{aligned}$$

The general solution of (†) is given by $x=\pm\frac{\pi}{3}+K\cdot 2\pi$ where $K\in\mathbb{Z}$, or $x=\frac{\pi}{2}+M\pi$ where $M\in\mathbb{Z}$.

Define

$$\begin{aligned} A &= \Big\{x\in\mathbb{R} : x=\frac{\pi}{3}+K\cdot 2\pi \text{ for some } K\in\mathbb{Z}\Big\}, \\ B &= \Big\{x\in\mathbb{R} : x=-\frac{\pi}{3}+K\cdot 2\pi \text{ for some } K\in\mathbb{Z}\Big\}, \\ C &= \Big\{x\in\mathbb{R} : x=\frac{\pi}{2}+M\pi \text{ for some } M\in\mathbb{Z}\Big\} \end{aligned}$$

The solution set of (†) is given by $A\cup B\cup C$.

(d) **Solution.**

Note that $6^2+8^2=100=10^2$. Take $\alpha=\arcsin(\frac{4}{5})$. Then $10\sin(\alpha)=8$, $10\cos(\alpha)=10\sqrt{1-\sin^2(\alpha)}=6$.

We proceed to solve the equation (†):

$$\begin{aligned} 6\sin(x)+8\cos(x) &= 5 \quad \text{——} \quad (\dagger) \\ 10\sin(x)\cos(\alpha)+10\cos(x)\sin(\alpha) &= 5 \\ 10\sin(x+\alpha) &= 5 \\ \sin(x+\alpha) &= \frac{1}{2} \\ x+\alpha &= (-1)^N\cdot\frac{\pi}{6}+N\pi \quad \text{where } N\in\mathbb{Z} \\ x &= \alpha+(-1)^N\cdot\frac{\pi}{6}+N\pi \end{aligned}$$

The general solution of (†) is given by $x=\alpha+(-1)^N\cdot\frac{\pi}{6}+N\pi$ where $N\in\mathbb{Z}$.

The solution set of (†) is given by $\Big\{x\in\mathbb{R} : x=\alpha+(-1)^N\cdot\frac{\pi}{6}+N\pi \text{ for some } N\in\mathbb{Z}\Big\}$.

(e) **Solution.**

We proceed to solve the equation (†):

$$\begin{array}{rcl} \tan(3\sqrt{x}) & = & 1 \quad \text{——} \quad (\dagger) \\ 3\sqrt{x} & = & \dfrac{\pi}{4} + N \cdot \pi \quad \text{where } N \in \mathbb{N} \\ x & = & \left(\dfrac{1}{4} + N\right)^2 \cdot \dfrac{\pi^2}{9} \end{array}$$

The general solution of (†) is given by $x = \left(\frac{1}{4} + N\right)^2 \cdot \frac{\pi^2}{9}$, where N is a natural number.

The solution set of (†) is given by $\left\{x \in \mathbb{R} : x = \left(\frac{1}{4} + N\right)^2 \cdot \frac{\pi^2}{9} \text{ for some } N \in \mathbb{N}\right\}$.

(f) **Solution.**

We proceed to solve the equation (†):

$$\begin{array}{rcl} \cos(\dfrac{1}{2x}) & = & 1 \quad \text{——} \quad (\dagger) \\ \dfrac{1}{2x} & = & 2N \cdot \pi \quad \text{where } N \in \mathbb{Z}\backslash\{0\} \\ x & = & \dfrac{1}{4N \cdot \pi} \end{array}$$

The general solution of (†) is given by $x = \frac{1}{4N \cdot \pi}$ where $N \in \mathbb{Z}\backslash\{0\}$.

The solution set of (†) is given by $\left\{x \in \mathbb{R} : x = \frac{1}{4N \cdot \pi} \text{ for some } N \in \mathbb{Z}\backslash\{0\}\right\}$.

6. **Solution.**

(a) Let $z = \dfrac{2(\sqrt{3} - i)}{\sqrt{3} + i}$.

Note that $z = \dfrac{2(\sqrt{3} - i)}{\sqrt{3} + i} = 2 \cdot \dfrac{(\sqrt{3} - i)/2}{(\sqrt{3} + i)/2} = 2 \cdot \dfrac{\cos(-\pi/6) + i\sin(-\pi/6)}{\cos(\pi/6) + i\sin(\pi/6)} = 2 \cdot \left(\cos\left(-\dfrac{\pi}{3}\right) + i\sin\left(-\dfrac{\pi}{3}\right)\right)$.

(b) i. We have $|z|^6 = 2^6$. Note that $2020 = 6 \cdot 336 + 4$. Then

$$\begin{array}{rcl} z^{2020} = z^{6 \cdot 336 + 4} = (z^6)^{336} \cdot z^4 = \left(2^6\right)^{336} \cdot 2^4 \cdot \cos\left(-4 \cdot \dfrac{\pi}{3}\right) + i\sin\left(-4 \cdot \dfrac{\pi}{3}\right) & = & 2^{2020}\left(\cos\left(\dfrac{2\pi}{3}\right) + i\sin\left(\dfrac{2\pi}{3}\right)\right) \\ & = & -2^{2019} + 2^{2019} \cdot \sqrt{3}i \end{array}$$

ii. The square roots of z are $\pm\sqrt{2} \cdot \left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$ respectively.

Expressed in standard form, they are $\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$, $-\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$ respectively.

7. (a) **Solution.**

We proceed to solve the equation (⋆):

$$\begin{array}{rcl} z^5 - 32i & = & 0 \quad \text{——} \quad (\star) \\ z^5 & = & 32i \\ \left(\dfrac{z}{2i}\right)^5 & = & 1 \\ \dfrac{z}{2i} & = & \omega^N, \text{ where } N = 0, 1, 2, 3, 4. \end{array}$$

Here ω stands for the number $\cos(\frac{2\pi}{5}) + i\sin(\frac{2\pi}{5})$.

So the general solution of the equation ($\star$) is given by $z = 2(\cos(\frac{\pi}{2} + N \cdot \frac{2\pi}{5}) + i\sin(\frac{\pi}{2} + N \cdot \frac{2\pi}{5}))$, where $N = 0, 1, 2, 3, 4$.

(b) **Answer.**

$z = 2(\cos(\frac{\pi}{10}) + i\sin(\frac{\pi}{10})$ or $z = 2(\cos(-\frac{3\pi}{10}) + i\sin(-\frac{3\pi}{10})$ or $z = 2(\cos(-\frac{7\pi}{10}) + i\sin(-\frac{7\pi}{10}))$.

8. (a) *Hint.* You need De Moivre's Theorem. Also recall the for any $\eta \in \mathbb{C}$, we have $2\mathsf{Re}(\eta) = \eta + \bar{\eta}$ and $2i\mathsf{Im}(\eta) = \eta - \bar{\eta}$.

   (b) *Hint.* You need the Binomial Theorem.

9. **Answer.**

   (a) (I) $x \in A$
   (II) There exists some
   (III) $x = 16m^6$
   (IV) $2, m \in \mathbb{Z}$
   (V) $x = 2(8m^6) = 2(2m^2)^3 = 2n^3$
   (VI) $x \in B$

   (b) (I) $x_0 = 2 \cdot 1^3$
   (II) 1
   (III) $x_0 \in B$
   (IV) Suppose it were true that $x_0 \in A$.
   (V) there would exist some $m \in \mathbb{Z}$
   (VI) $2 \cdot (4m^6) = 8m^6 = 1$
   (VII) $4m^6 \in \mathbb{Z}$
   (VIII) divisible

10. **Solution.**

Let $A = \{x \mid x = r^6 \text{ for some } r \in \mathbb{Q}\}$, $B = \{x \mid x = r^2 \text{ for some } r \in \mathbb{Q}\}$.

   (a) Pick any $x \in A$.
   There exists some $r \in \mathbb{Q}$ such that $x = r^6$.
   Take $s = r^3$. Note that $s \in \mathbb{Q}$. We have $x = (r^3)^2 = s^2$.
   Hence $x \in B$.
   It follows that $A \subset B$.

   (b) Let $x_0 = 4$.
   Note that $x_0 = 2^2$ and $2 \in \mathbb{Q}$. Then $x_0 \in B$.
   We verify that $x_0 \notin A$:

   - Suppose it were true that $x_0 \in A$.
     Then there would exist some $r \in \mathbb{Q}$ such that $x_0 = r^6$.
     Now $4 = r^6$.
     Since $r \in \mathbb{R}$, we would have $r = \sqrt[3]{2}$ or $r = -\sqrt[3]{2}$.
     But $\sqrt[3]{2}, -\sqrt[3]{2} \notin \mathbb{Q}$. Contradiction arises.
     Therefore the assumption that $x_0 \in A$ is false. We have $x_0 \notin A$.

   Hence $B \not\subset A$.

11. **Solution.**

Let $C = \{\zeta \in \mathbb{C} : |\zeta - 1| \leq 1\}, D = \{\zeta \in \mathbb{C} : |\zeta| \leq 2\}$.

   (a) Pick any $\zeta \in C$.
   We have $|\zeta - 1| \leq 1$ (by the definition of C).
   By the Triangle Inequality, we have $|\zeta| - 1 \leq |\zeta - 1| \leq 1$.
   Then $|\zeta| \leq 2$. Therefore, we have $\zeta \in D$ (by the definition of D).
   It follows that $C \subset D$.

(b) Take $\zeta_0 = -1$. We have $\zeta_0 \in \mathbb{C}$.
Note that $|\zeta_0| = 1 \leq 2$. Then $\zeta_0 \in D$.
We verify that $\zeta_0 \notin C$:
  - We have $|\zeta_0 - 1| = |-1-1| = 2 > 1$. Then $\zeta_0 \notin C$.

It follows that $D \not\subset C$.

12. **Answer.**
   (a) (I) $A \cup B \subset B$
   (II) it were true that $A \backslash B \neq \emptyset$
   (III) $x_0 \in A \backslash B$
   (IV) $x_0 \in A$
   (V) $x_0 \in B$
   (VI) $x_0 \in A \cup B$
   (VII) $x_0 \in B$
   (VIII) Contradiction arises
   (b) ——

13. (a) **Solution.**
   Let A, B, C, D be sets. Suppose $A \subset C$ and $B \subset D$.
   Pick any object x.
   Suppose $x \in A \cup B$. Then $x \in A$ or $x \in B$.
   (Case 1). Suppose $x \in A$. Then, since $A \subset C$, we have $x \in C$. Therefore $x \in C$ or $x \in D$. Hence $x \in C \cup D$.
   (Case 2). Suppose $x \notin A$. Then $x \in B$. Therefore, since $B \subset D$, we have $x \in D$. Then $x \in C$ or $x \in D$. Hence $x \in C \cup D$.
   Hence in any case, we have $x \in C \cup D$.
   It follows that $A \cup B \subset C \cup D$.
   (b) ——
   (c) ——

14. ——

15. **Answer.**
   (a) False. (b) True. (c) False. (d) False. (e) True.

16. **Solution.**
   Consider the predicate '$x \notin x$', which we denote by $P(x)$.
   (a) Denote by R the object $\{x \mid P(x)\}$. Suppose it were true that R was a set.
      i. Suppose it were true that the object R is an element of the set R. Then $R \in R$. By the definition of R, $P(R)$ is a true statement. Then $R \notin R$. We have $R \in R$ and $R \notin R$. Contradiction arises.
      Hence the object R is not an element of the set R.
      ii. Suppose it were true that the object R is not an element of the set R. Then $R \notin R$. Therefore, $P(R)$ is a true statement. Hence, by the definition of R, $R \in R$. We have $R \notin R$ and $R \in R$. Contradiction arises.
      Hence the object R is an element of the set R.

   Remark. Hence the object R is not a set in the first place. The failure of the construction $\{x \mid x \notin x\}$ in giving a set is known as **Russell's Paradox**.
   (b) Let A be a set. Denote by B the object $\{x \in A : P(x)\}$.
   Suppose B were an element of A. Then $B \in A$. We have $B \in B$ or $B \notin B$.
      - (Case 1). Suppose $B \in B$. Then, by the definition of B, $P(B)$ is a true statement. Therefore $B \notin B$. We have $B \in B$ and $B \notin B$. Contradiction arises.
      - (Case 2). Suppose $B \notin B$. Then $P(B)$ is a true statement, Therefore, by the definition of B, $B \in B$. We have $B \notin B$ and $B \in B$. Contradiction arises.

   Hence, in every case, contradiction arises. B is not an element of A in the first place.