MATH1050 Exercise 6

1. Let P, Q, R be statements. Consider each of the statements below. Determine whether it is a tautology or a contradiction or a contingent statement . Justify your answer by drawing an appropriate truth table.

   (a) $(P \to R) \to [(P \to Q) \wedge (Q \to R)]$

   (b) $(P \to Q) \to [(P \to R) \vee (Q \to R)]$

2. *You are not required to justify your answer.*

   Let $C = \{\{0,1\},\{1\},\{1,2,3\},\{3,4\}\}$, $D = \{\{0,1,1\},\{1,2,3\},\{\{3\},\{4\}\}\}$.

   Consider each of the sets below. List every element of the set concerned, each element exactly once. Where the set concerned is the empty set, write '*this set is the empty set*'.

   (a) $C \cap D$.

   (b) $C \cup D$.

   (c) $C \backslash D$.

   (d) $C \triangle D$.

   (e) $\mathfrak{P}(C \backslash D)$.

3. *You are not required to justify your answers in this question.*

   Let $M = \{m,a,r,c,u,s\}$, $T = \{t,u,l,l,i,u,s\}$, $C = \{c,i,c,e,r,o\}$.

   (a) How many elements are there in the set C?

   (b) How many elements are there in the set $M \cup T$?

   (c) How many elements are there in the set $(M \cup T)\backslash C$?

   (d) How many elements are there in the set $\{(M \cup T)\backslash C\}$?

   (e) How many elements are there in the set $(\{M\} \cup \{T\})\backslash\{C\}$?

   (f) How many elements are there in the set $\{M \cup T\}\backslash\{C\}$?

   (g) List every element of the set $M \cap C$, each element exactly once.

   (h) List every element of the set $\mathfrak{P}(M \cap C)$, each element exactly once.

4. *You are not required to justify your answers in this question.*

   Let $A = \{x \in \mathbb{N}\backslash\{0,1,2,3,4,5\} : \sqrt{x} = k \cdot \sqrt{5} \text{ for any } k \in \mathbb{N}\}$, $B = \{x \in \mathbb{N}\backslash\{0,1,2,3,4,5\} : \sqrt{x} = k \cdot \sqrt{5} \text{ for some } k \in \mathbb{N}\}$.

   (a) Is A the empty set? If *yes*, just write '$A = \emptyset$'. If *no*, write '$A \neq \emptyset$' and also name one element of A.

   (b) Is B the empty set? If *yes*, just write '$B = \emptyset$'. If *no*, write '$B \neq \emptyset$' and also name one element of B.

5. For each equation with unknown in the reals below, determine its solution set by solving for its general solution.

   You are not required to give the 'checking step' explicitly, but be careful not to wrongly include false candidates amongst the solution, nor wrongly ignore a genuine solution.

   (a) $\cos(2x) = \sin(x)$.

   (b) $\sin(2x) + \sin(8x) = \sin(5x)$.

   (c) $2\sin(\frac{x}{2})\sin(\frac{3x}{2}) = 1$.

   (d) $6\sin(x) + 8\cos(x) = 5$.

   (e)$^\diamond$ $\tan(3\sqrt{x}) = 1$.

   (f)$^\diamond$ $\cos(\frac{1}{2x}) = 1$.

6. Let $z = \dfrac{2(\sqrt{3} - i)}{\sqrt{3} + i}$.

(a) Express z in polar form.

(b) Hence, or otherwise, find the numbers below. Express the respective answers in standard form.

i. z^{2020}.

ii. The square roots of z.

7. (a) Solve for all solutions of the equation $z^5 - 32i = 0$ with complex unknown z.
Express your answer in polar form.

(b) *In this part, there is no need to justify your answer.*
Write down all the complex solutions of the system of inequalities

$$\begin{cases} z^5 - 32i & = & 0 \\ \mathsf{Re}(z) & \geq & \mathsf{Im}(z) \end{cases}$$

8.◇ Let $\theta \in \mathbb{R}$, and $z = \cos(\theta) + i\sin(\theta)$.

(a) Let $m \in \mathbb{N}\backslash\{0\}$. Prove that $2\cos(m\theta) = z^m + \overline{z}^m$ and $2i\sin(m\theta) = z^m - \overline{z}^m$.

(b) Let $n \in \mathbb{N}\backslash\{0\}$. Applying the result above, or otherwise, prove the statements below:

i. $\cos(2n\theta) = \sum_{j=0}^{n}(-1)^j \binom{2n}{2j} \cos^{2n-2j}(\theta)\sin^{2j}(\theta)$, *and*

$\sin(2n\theta) = \sum_{j=0}^{n-1}(-1)^j \binom{2n}{2j+1} \cos^{2n-2j-1}(\theta)\sin^{2j+1}(\theta)$.

ii. $\cos((2n+1)\theta) = \sum_{j=0}^{n}(-1)^j \binom{2n+1}{2j} \cos^{2n+1-2j}(\theta)\sin^{2j}(\theta)$, *and*

$\sin((2n+1)\theta) = \sum_{j=0}^{n}(-1)^j \binom{2n+1}{2j+1} \cos^{2n-2j}(\theta)\sin^{2j+1}(\theta)$.

9. Let $A = \{x \mid x = 16m^6 \text{ for some } m \in \mathbb{Z}\}$, $B = \{x \mid x = 2m^3 \text{ for some } m \in \mathbb{Z}\}$.

(a) Consider the statement (S):

(S) $A \subset B$.

Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate passages so that it gives an argument for the statement (S).

Pick any ____(I)____ .
____(II)____ $m \in \mathbb{Z}$ such that ____(III)____ .
Define $n = 2m^2$. Since ____(IV)____ , we have $n \in \mathbb{Z}$.
We have ____(V)____ .
Hence ____(VI)____ .
It follows that $A \subset B$.

(b) Consider the statement (N):

(N) $B \not\subset A$.

Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate passages so that it gives an argument for the statement (N).

Let $x_0 = 2$.
Note that ______(I)______ and ___(II)___ $\in \mathbb{Z}$.
Then ______(III)______ .
We verify that $x_0 \notin A$:

- ______________(IV)______________

  Then __________(V)__________ such that $x_0 = 16m^6$.

  Now $2 = x_0 = 16m^6$.

  Then __________(VI)__________ .

  Since $4, m \in \mathbb{Z}$, we have ________(VII)________ .

  Then 1 would be ___(VIII)___ by 2. Contradiction arises.

  Therefore the assumption that $x_0 \in A$ is false. We have $x_0 \notin A$.

Hence $B \not\subset A$.

10. Let $A = \{x \mid x = r^6 \text{ for some } r \in \mathbb{Q}\}$, $B = \{x \mid x = r^2 \text{ for some } r \in \mathbb{Q}\}$.

    (a) Prove that $A \subset B$.

    (b)$^\diamondsuit$ Determine whether it is true that $B \subset A$. Justify your answer.

11. Let $C = \{\zeta \in \mathbb{C} : |\zeta - 1| \leq 1\}, D = \{\zeta \in \mathbb{C} : |\zeta| \leq 2\}$.

    (a) Prove that $C \subset D$.
    Remark. Make good use of the Triangle Inequality for complex numbers.

    (b) Prove that $D \not\subset C$.

12. (a) Consider the statement (P):

    (P) *Let A, B be sets. Suppose $A \cup B \subset B$. Then $A \backslash B = \emptyset$.*

    Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate passages so that it gives a proof-by-contradiction argument for the statement (P).

    Let A, B be sets. Suppose ______(I)______ . Further suppose __________(II)__________ .
    Take some ______(III)______ . We would have $x_0 \in A$ and $x_0 \notin B$ by definition of complement.
    In particular $x_0 \in A$. . Then ______(IV)______ or ______(V)______ .
    Therefore ______(VI)______ by definition of union.
    Since $x_0 \in A \cup B$ and $A \cup B \subset B$, we would have ______(VII)______ .
    Now we have $x_0 \in B$ and $x_0 \notin B$. ______(VIII)______ .
    It follows that $A \backslash B = \emptyset$ in the first place.

    (b) Prove the statements below 'from first principles', using the definitions of set equality, subset relation, intersection, union, complement, where appropriate:

    i. *Let A, B be sets. Suppose $A \subset A \backslash B$. Then $A \cap B = \emptyset$.*
    ii. *Let A, B be sets. Suppose $A \subset A \cap B$. Then $A \backslash B = \emptyset$.*

    Remark. It is advisable to use the proof-by-contradiction method. (Why? Think about what you need do, according to the definition of set equality, in order to establish the equality in the conclusion of the statement.)

13. Prove each of the statements below 'from first principles', using the definitions of set equality, subset relation, intersection, union, complement, where appropriate.

    (a) *Let A, B, C, D be sets. Suppose $A \subset C$ and $B \subset D$. Then $A \cup B \subset C \cup D$.*

    (b) *Let A, B, C be sets. Suppose $A \subset B$, $B \subset C$, and $C \subset A$. Then $A = B$.*

(c)$^\diamondsuit$ *Let A, B, C be sets. Suppose $A \subset C$ and $B \subset C$. Then $(C \backslash A) \backslash (C \backslash B) = B \backslash A$.*

14. Prove the statements below 'from first principles', using the definitions of set equality, subset relation, intersection, union, complement, where appropriate:

    (a) *Let a, b be two objects (not necessarily distinct). $\{a\} = \{b\}$ iff $a = b$.*

    (b) *Let a, b, c be three objects (not necessarily distinct). $\{a, b\} = \{c\}$ iff $a = b = c$.*

    (c)$^\clubsuit$ *Let a, b, c, d be four objects (not necessarily distinct). $\{a, b\} = \{c, d\}$ iff (($a = c$ and $b = d$) or ($a = d$ and $b = c$) or $a = b = c = d$).*

    (d)$^\diamondsuit$ *Let A, B, C, D be sets. Suppose $\{A, B\} = \{C, D\}$. Then $A \cap B = C \cap D$ and $A \cup B = C \cup D$.*

    Remark. It might help to recall that according to the Method of Specification, $\{p\} = \{x \mid x = p\}$, and $\{p, q\} = \{x \mid x = p \text{ or } x = q\}$ *et cetera.*

15.$^\diamondsuit$ In this question, you may assume the validity of the statements without proof:

    (♯) *Let a, b be two objects (not necessarily distinct). $\{a\} = \{b\}$ iff $a = b$.*

    (♭) *Let a, b, c be three objects (not necessarily distinct). $\{a, b\} = \{c\}$ iff $a = b = c$.*

    (♮) $\emptyset \neq \{\emptyset\}$.

    Let $A = \{\emptyset\}$, $B = \{\{\emptyset\}\}$, $C = \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$, $D = \{\emptyset, \{\{\emptyset\}\}\}$. For each of the statements below, determine whether it is true or false. Justify your answer with an appropriate argument:

    (a) $A \in C$.

    (b) $A \subset C$.

    (c) $B \subset D$.

    (d) $B \in C$.

    (e) $A \cup B \in C$.

    (f) $C \cap D = \emptyset$.

16.$^\diamondsuit$ Denote by $P(x)$ the predicate '$x \notin x$' (with variable x).

    (a) Denote by R the object $\{x \mid P(x)\}$, obtained from the Method of Specification. (R is called the '**Russell set**'.)

    Suppose it were true that R was a set. (Hence it makes sense to discuss whether an arbitrarily given object is an element of R or not.)

    i. Can it happen that the object R is an element of the set R? Why?

    ii. Can it happen that the object R is not an element of the set R? Why?

    Remark. From the answers to the above questions, you would have to conclude that R is not a set in the first place. (Why?) This tells us the above application of the Method of Specification leads to a serious trouble: the construction $\{x \mid P(x)\}$ fails to give a set. This trouble is known as **Russell's Paradox**.

    (b) Let A be a set. Denote by B the object $\{x \in A : P(x)\}$, obtained from the Method of Specification. (This time it is guaranteed that B is a set, because we are constructing a subset from the given set A.)

    Prove that B is not an element of A. (Apply the proof-by-contradiction method.)

    Remark. This shows that given any set, there is always some object which does not belong to it as an element. Hence no set contains every conceivable object as its element. There is no such thing as 'universal set'.