MATH1050 Exercise 5 Supplement

1. Let n be an integer greater than 1.

   (a) Prove that $\binom{2n-1}{n-1} - \binom{2n-1}{n-2} = \frac{(2n)!}{(n!)[(An+B)!]}$. Here A, B are appropriate positive integers whose respective values you have to determine explicitly.

   (b) Hence, or otherwise prove that $\binom{2n}{n}$ is divisible by $n+1$.

2. Let n be a positive integer.

   (a) Prove that $2\binom{3n+1}{n} - \binom{3n+1}{n+1} = \frac{(3n+1)!}{[(n+A)!][(2n+B)!]}$. Here A, B are appropriate positive integers whose respective values you have to determine explicitly.

   (b) Hence, or otherwise prove that $\binom{3n+1}{n}$ is divisible by $n+1$, and $\binom{3n+1}{n+1}$ is divisible by $2n+1$.

3.♣ Let a be a real number, n be a positive integer, and $f(x)$ be the polynomial given by $f(x) = (1 + x + ax^2)^{6n}$. Denote the coefficients of the x-term, the x^2-term, and the x^3-term in the polynomial $f(x)$ by k_1, k_2, k_3 respectively.

   (a) Express k_1, k_2, k_3 in terms of a.

   (b) Suppose k_1, k_2, k_3 are in arithmetic progression.

      i. Prove that $a = \frac{An^2 + Bn + C}{9(2n-1)}$. Here A, B, C are some appropriate integers whose values you have to determine explicitly.

      ii. Further suppose $a \geq 0$. What is the value of n? Justify your answer.

4.♢ Let m, n be positive integers. Suppose $m > n$. Let $f(x)$ be the polynomial given by $f(x) = (1+x)^{mn}(1-x)^{m(n-1)}$. Prove that the coefficients of the x-term and the x^2-term are equal to each other iff $m = 2n+1$.

5. Apply mathematical induction to prove the statements below:

   (a) $\sum_{k=2}^{n} \binom{n}{2} = \binom{n+1}{3}$ *for any integer greater than* 1.

   (b) $n! < \left(\frac{n}{2}\right)^n$ *for any integer greater than* 5.

   (c) $\frac{2^{2n}}{2n} < \binom{2n}{n} < \frac{2^{2n}}{4}$ *for any integer greater than* 7.

6. Prove the statement below:

   - *Let a, n be positive integers. Suppose $n \geq a$. Then $(2a-1)^n + (2a)^n < (2a+1)^n$.*

   Remark. There is no need to apply mathematical induction.

7.♢ Let m be a positive integer. Prove that $\sum_{k=0}^{m} 2^{2k} \binom{2m}{2k} = \frac{A^m + B}{2}$. Here A, B are some positive integers whose respective values you have to determine explicitly.

8.♢ Prove the statement below, which is known as **Vandemonde's Theorem**:

   - *Let p, q, r be non-negative integers. Suppose $r \leq p+q$. Then* $\sum_{k=0}^{r} \binom{p}{k}\binom{q}{r-k} = \binom{p+q}{r}$.

   (*Hint.* Note that $(1+x)^{p+q} = (1+x)^p(1+x)^q$ as polynomials.)

9.♢ Let n be a positive integer. Find the respective values of the numbers below. Leave your answer in terms of n.

(a) $\sum_{k=0}^{n} \binom{n}{k}^2$.

(b) $\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2$.

10. Let n be a positive integer, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = (1+x)^n$ for any $x \in \mathbb{R}$.

    (a) Suppose $n \geq 3$. By differentiating f, or otherwise, prove that $\sum_{k=0}^{n} \frac{k(k-1)(k-2)}{3^k} \binom{n}{k} = \frac{n(n-1)(n-A) \cdot B^{n-C}}{3^n}$.

    Here A, B, C are some appropriate integers whose respective values you have to determine explicitly.

    (b) By integrating f, or otherwise, prove that $\sum_{k=0}^{n} \frac{2^k}{(k+3)(k+2)(k+1)} \binom{n}{k} = \frac{A^{n+3} - 1 - 2(n+B)^2}{C(n+3)(n+2)(n+1)}$.

    Here A, B, C are some appropriate integers whose respective values you have to determine explicitly.

11. (a) Let n, m be positive integers.

    i.$^\diamondsuit$ Verify the equality $x[(1+x)^n + (1+x)^{n+1} + \cdots + (1+x)^{n+m}] = (1+x)^{n+m+1} - (1+x)^n$ for polynomials.

    ii.$^\clubsuit$ Let k be a positive integer. Write $c_{n,m,k} = \binom{n}{k} + \binom{n+1}{k} + \binom{n+2}{k} + \cdots + \binom{n+m}{k}$.

    A. Suppose $k < n$. What is the value of $c_{n,m,k}$? Leave your answer in terms of n, m, k where appropriate.

    B. Suppose $n \leq k \leq n+m$. What is the value of $c_{n,m,k}$? Leave your answer in terms of n, m, k where appropriate.

    (b) Let m be a positive integer.

    i.$^\clubsuit$ Applying the results in the previous parts, or otherwise, prove that

$$\sum_{r=5}^{m+4} r(r-1)(r-2)(r-3) = 24(\binom{m+5}{5} - 1).$$

    ii. Hence, or otherwise, find the value of $\sum_{r=0}^{m+4} r(r-1)(r-2)(r-3)$. Leave your answer in terms of m where appropriate.

12. For each $n \in \mathbb{N}\backslash\{0,1\}$, define $a_n = \sqrt[n]{n} - 1$.

    (a) Prove that $a_n \geq 0$ for any $n \in \mathbb{N}\backslash\{0,1\}$.

    (b) By applying the Binomial Theorem to the expression $(1+a_n)^n$, prove that $a_n \leq \sqrt{\frac{2}{n-1}}$ for any $n \in \mathbb{N}\backslash\{0,1\}$.

    Remark. The inequalities described here constitute the key step in the argument for the statement '$\lim_{n\to\infty} \sqrt[n]{n} = 1$'.

13. *Familiarity with the calculus of one variable is assumed in this question.*

    Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = e^{x^2/2}$ for any $x \in \mathbb{R}$.

    Take for granted that the exponential function $\exp : \mathbb{R} \longrightarrow \mathbb{R}$ *is differentiable on* $\mathbb{R}$*, and every polynomial function is differentiable on* $\mathbb{R}$.

    (a) Verify that $f'(x) = xf(x)$ for any $x \in \mathbb{R}$.

    (b)$^\clubsuit$ Apply mathematical induction to prove the statement ($\sharp$):

    ($\sharp$) *Let* $n \in \mathbb{N}\backslash\{0\}$. *The function* f *is* $(n+1)$*-times differentiable, and for any* $x \in \mathbb{R}$,

$$f^{(n+1)}(x) = xf^{(n)}(x) + nf^{(n-1)}(x).$$

    (c)$^\heartsuit$ Apply mathematical induction to prove the statement ($\flat$):

    ($\flat$) *Let* $n \in \mathbb{N}\backslash\{0\}$. *There exists some polynomial function* P_n *of degree* n *and with leading coefficient* 1 *such that* $f^{(n)}(x) = P_n(x)e^{x^2/2}$ *for any* $x \in \mathbb{R}$.

    (d)$^\diamondsuit$ Prove that $f^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{n!}{2^{n/2}[(n/2)!]} & \text{if } n \text{ is even} \end{cases}$

14. *Familiarity with the calculus of one variable is assumed in this question.*

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{1}{1+x^2}$ for any $x \in \mathbb{R}$.

Take for granted that f is smooth on $\mathbb{R}$.

(a) i. By applying mathematical induction, or Leibniz's Rule, prove that for any $n \in \mathbb{N}$, for any $x \in \mathbb{R}$,

$$(1+x^2)f^{(n+2)}(x) + 2(n+2)xf^{(n+1)}(x) + (n+2)(n+1)f^{(n)}(x) = 0.$$

ii. Determine the value of $f^{(n)}(0)$ for each n.

(b) For each $n \in \mathbb{N}$, define the function $g_n : \mathbb{R} \longrightarrow \mathbb{R}$ by $g_n(x) = (1+x^2)^{n+1}f^{(n)}(x)$ for any $x \in \mathbb{R}$. *Take for granted that g_n is smooth on $\mathbb{R}$ for each n.*

i. Applying the results above, or otherwise, prove that for any $n \in \mathbb{N}$, for any $x \in \mathbb{R}$,

$$g_{n+2}(x) + 2(n+2)xg_{n+1}(x) + (n+2)(n+1)(1+x^2)g_n(x) = 0.$$

ii.$^\diamondsuit$ Hence, or otherwise, deduce that for any $n \in \mathbb{N}$, for any $x \in \mathbb{R}$,

$$(1+x^2)g_n''(x) - 2nxg_n'(x) + n(n+1)g_n(x) = 0.$$

iii.$^\clubsuit$ Applying mathematical induction, or otherwise, prove that for each $n \in \mathbb{N}$, g_n is a polynomial function of degree n and with leading coefficient $(-1)^n[(n+1)!]$.

15. *Familiarity with the calculus of one variable is assumed in this question.*

For each $n \in \mathbb{N}$, define the function $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ by $f_n(x) = x^n|x|$ for any $x \in \mathbb{R}$.

Take for granted that f_n is smooth at every point in $\mathbb{R}\backslash\{0\}$. (The point in this question is the behaviour of the function f_n at and near 0.)

(a) i. Verify that f_0 is continuous at 0.

ii. Verify that f_0 is not differentiable at 0.

(b) Verify that for any $n \in \mathbb{N}\backslash\{0\}$, the function f_n is differentiable at 0, and $f_n'(0) = 0$.

(c) Verify that for each $n \in \mathbb{N}\backslash\{0\}$, $f_n'(x) = (n+1)f_{n-1}(x)$ for any $x \in \mathbb{R}$.

(d)$^\diamondsuit$ By applying the Telescopic Method, or otherwise, prove that for any $n \in \mathbb{N}\backslash\{0\}$, for each $k = 1, 2, \cdots, n$, there exists some $A_{n,k} \in \mathbb{R}$ such that $f_n^{(k)}(x) = A_{n,k}f_{n-k}(x)$ for any $x \in \mathbb{R}\backslash\{0\}$.

(e)$^\diamondsuit$ Prove that for any $n \in \mathbb{N}\backslash\{0\}$, the function f_n is n-times continuously differentiable at 0, and $f_n^{(k)}(0) = 0$ for each $k = 1, 2, \cdots, n$.

Remark. Proceed as described here:

Let $n \in \mathbb{N}\backslash\{0\}$. Denote by $Q(k)$ the proposition below:

f_n is k-times continuously differentiable at 0, and $f_n^{(k)}(0) = 0$.

First verify that $Q(0)$ is true. Next, verify that for each $k = 1, 2, \cdots, n-1$, if $Q(k)$ is true then $Q(k+1)$ is true.

It will then follow (from a repeated application of modus ponens and hypothetical syllogism) that each of $Q(0), Q(1), Q(2), \cdots, Q(n)$ are all true.

Such an argument is referred to as **finite induction**.

(f) Prove that f_n is not $(n+1)$-times differentiable at 0 for each $n \in \mathbb{N}\backslash\{0\}$.

16. *Familiarity with the calculus of one variable is assumed in this question.*

For each $n \in \mathbb{N}$, define the function $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f_n(x) = \begin{cases} -x^n \ln(x^2) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Take for granted the result $\lim\limits_{x\to 0^\pm} f_0(x) = +\infty$. Also take for granted the result that f_n is smooth at every point of $\mathbb{R}\backslash\{0\}$. (The point in this question is the behaviour of the function f_n at and near 0.)

(a) By applying L'Hôpital's Rule, or otherwise, verify that $\lim\limits_{x\to 0^\pm} f_n(x) = 0$ respectively for each $n \in \mathbb{N}\backslash\{0\}$. (There is no need to apply mathematical induction.)

(b) Prove that f_1 is continuous at 0 but not differentiable at 0.

(c) Suppose $n \in \mathbb{N} \backslash \{0,1\}$. Prove the statements below:

i. *f_n is differentiable at 0, and $f_n'(0) = 0$.*

ii. *$f_n'(x) = nf_{n-1}(x) - 2x^{n-1}$ for any $x \in \mathbb{R}$.*

iii. *f_n is continuously differentiable at 0.*

(d)♣ By applying the Telescopic Method, or otherwise, prove that for each $n \in \mathbb{N} \backslash \{0,1\}$, for each $k = 1, 2, \cdots, n-1$, there exists some $A_{n,k} \in \mathbb{R}$ such that $f_n^{(k)}(x) = \dfrac{n!}{(n-k)!} f_{n-k}(x) - A_{n,k}x^{n-k}$ for any $x \in \mathbb{R} \backslash \{0\}$.

(e)♣ Hence, or otherwise, deduce that for each $n \in \mathbb{N} \backslash \{0,1\}$, the function f_n is $(n-1)$-times continuously differentiable at 0, and $f_n^{(k)}(0) = 0$ for each $k = 1, 2, 3, \cdots, n-1$.

(f) Prove that f_n is not n-times differentiable at 0 for each $n \in \mathbb{N} \backslash \{0,1\}$.

17. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function. Suppose that $f(x+y) = f(x)f(y)$ for any $x, y \in \mathbb{R}$. Further suppose that f is not a constant function.

(a) Prove that $f(0) = 1$.

(b)◇ Prove that $f(x) \geq 0$ for any $x \in \mathbb{R}$.

(c) Prove that for any $x \in \mathbb{R}$, $f(x) > 0$ and $f(-x) > 0$ and $f(-x) = \dfrac{1}{f(x)}$.

(d) Prove that $f(nx) = (f(x))^n$ for any $n \in \mathbb{N}$ for any $x \in \mathbb{R}$.

(e) Prove that $f(mx) = (f(x))^m$ for any $m \in \mathbb{Z}$ for any $x \in \mathbb{R}$.

(f) Prove that $f(rx) = (f(x))^r$ for any $r \in \mathbb{Q}$, for any $x \in \mathbb{R}$.

(g) *Familiarity with the calculus of one variable is assumed in this part.*
Take for granted the validity of the results below:

- *For any $u \in \mathbb{R}$, there exists some infinite sequence of rational numbers $\{s_n\}_{n=0}^{\infty}$ such that $\lim\limits_{n \to \infty} s_n = u$.*

Now suppose f is continuous on $\mathbb{R}$.
Prove that there exists some positive real number c such that $f(x) = c^x$ for any $x \in \mathbb{R}$.

18. *Familiarity with the calculus of one variable is assumed in this question.*
Let $f : [0, +\infty) \longrightarrow \mathbb{R}$ be a continuous function. Suppose that for any $x \in [0, +\infty)$,

$$f(x) \geq 0 \quad \text{and} \quad f(x) \geq 1 + \int_0^x 2uf(u)du.$$

(a)♣ Apply mathematical induction to prove that for any $n \in \mathbb{N}$, $f(x) \geq \displaystyle\sum_{j=0}^{n} \frac{x^{2j}}{j!}$ for any $x \in [0, +\infty)$.

(b) Prove that $f(\sqrt{e}) \geq e^e$.