MATH1050 Exercise 5 (Answers and selected solution)

1. **Solution.**

(a) Let $n \in \mathbb{N}$, and $k \in \mathbb{Z}$.

i.
- (Case 1.) Suppose $0 \leq k \leq n$.
Then $\binom{n}{k} = \frac{n!}{k! \cdot [(n-k)!]} = \frac{n!}{[n-(n-k)!] \cdot [(n-k)!]} = \frac{n!}{[(n-k)!] \cdot [n-(n-k)!]} = \binom{n}{n-k}$.
- (Case 2.) Suppose $k < 0$. Then $n - k > n$. Therefore $\binom{n}{k} = 0 = \binom{n}{n-k}$.
- (Case 3.) Suppose $k > n$. Then $n - k < 0$. Therefore $\binom{n}{k} = 0 = \binom{n}{n-k}$.

ii.
- (Case 1.) Suppose $0 \leq k \leq n-1$. Then

$$\begin{aligned}
\binom{n}{k} + \binom{n}{k+1} &= \frac{n!}{k! \cdot (n-k)!} + \frac{n!}{(k+1)! \cdot (n-k-1)!} \\
&= \frac{n!}{k! \cdot (n-k-1)!} \cdot \left(\frac{1}{n-k} + \frac{1}{k+1} \right) \\
&= \frac{n!}{k! \cdot (n-k-1)!} \cdot \frac{n+1}{(n-k)(k+1)} \\
&= \frac{(n+1)!}{(k+1)! \cdot (n-k)!} = \frac{(n+1)!}{(k+1)! \cdot [(n+1)-(k+1)]!} = \binom{n+1}{k+1}.
\end{aligned}$$

- (Case 2.) Suppose $k < -1$. Then $k + 1 < 0$.
$\binom{n}{k} = \binom{n}{k+1} = 0 = \binom{n+1}{k+1}$. Therefore $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$.
- (Case 3.) Suppose $k = -1$. Then $k + 1 = 0$.
$\binom{n}{k} = 0$ and $\binom{n}{k+1} = 1 = \binom{n+1}{k+1}$. Therefore $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$.
- (Case 4.) Suppose $k \geq n$. Then $n - k \leq 0$. Also, $n - k - 1 \leq 0$

$$\begin{aligned}
\binom{n}{k} + \binom{n}{k+1} &= \binom{n}{n-k} + \binom{n}{n-k-1} \\
&= \binom{n}{(n-k-1)+1} + \binom{n}{n-k-1} \\
&= \binom{n+1}{(n-k-1)+1} = \binom{n+1}{(n+1)-(k+1)} = \binom{n+1}{k+1}
\end{aligned}$$

(b) i. Suppose $n, m \in \mathbb{N}$.

For each $j = 1, 2, \cdots, m$, we have $\binom{n+j+1}{j} - \binom{n+j}{j-1} = \binom{n+j}{j}$.

Then

$$\begin{aligned}
\sum_{j=0}^{m} \binom{n+j}{j} &= \binom{n}{0} + \sum_{j=1}^{m} \binom{n+j}{j} \\
&= \binom{n}{0} + \sum_{j=1}^{m} (\binom{n+j+1}{j} - \binom{n+j}{j-1}) \\
&= \binom{n}{0} + (\binom{n+m+1}{m} - \binom{n+1}{1-1}) \\
&= \binom{n+m+1}{m}
\end{aligned}$$

ii. Suppose $n, m \in \mathbb{N}$.

For each $j = 1, 2, \cdots, m$, we have $\binom{n+j+1}{n+1} - \binom{n+j}{n+1} = \binom{n+j}{n}$

Then

$$
\begin{aligned}
\sum_{j=0}^{m}\binom{n+j}{n} &= \binom{n}{n}+\sum_{j=1}^{m}\binom{n+j}{n} \\
&= \binom{n}{n}+\sum_{j=1}^{m}(\binom{n+j+1}{n+1}-\binom{n+j}{n+1}) \\
&= \binom{n}{n}+(\binom{n+m+1}{n+1}-\binom{n+1}{n+1}) \\
&= \binom{n+m+1}{n+1}
\end{aligned}
$$

2. **Solution.**

   Let n be a positive integer.

   (a) Suppose r is an integer amongst $0, 1, \cdots, n$.

   Then $\binom{n+1}{r+1} / \binom{n+1}{r} = \dfrac{(n+1)!}{[(r+1)!]\{[(n+1)-r-1]!\}} \cdot \dfrac{(r!)\{[(n+1)-r]!\}}{(n+1)!} = \cdots = \dfrac{n+1-r}{r+1}$.

   (b) i.

$$
\begin{aligned}
\sum_{k=0}^{n}(k+1)\cdot\binom{n+1}{k+1}/\binom{n+1}{k} &= \sum_{k=0}^{n}(k+1)\cdot\frac{n+1-k}{k+1} = \sum_{k=0}^{n}(n+1-k) \\
&= \sum_{k=0}^{n}(n+1)-\sum_{k=0}^{n}k \\
&= (n+1)^2-\frac{(n+1)n}{2} = \frac{n^2+3n+2}{2}
\end{aligned}
$$

   ii.

$$
\begin{aligned}
\prod_{k=0}^{n}\left(\binom{n+1}{k+1}+\binom{n+1}{k}\right) &= \prod_{k=0}^{n}\left[\left(\binom{n+1}{k+1}/\binom{n+1}{k}+1\right)\cdot\binom{n+1}{k}\right] \\
&= \prod_{k=0}^{n}\left[\left(\frac{n+1-k}{k+1}+1\right)\cdot\binom{n+1}{k}\right] \\
&= \prod_{k=0}^{n}\left[\frac{n+2}{k+1}\cdot\binom{n+1}{k}\right] \\
&= \left[\prod_{k=0}^{n}(n+2)\right]\left[\prod_{k=0}^{n}\frac{1}{k+1}\right]\cdot\left(\prod_{k=0}^{n}\binom{n+1}{k}\right) \\
&= \frac{(n+2)^{n+1}}{[(n+1)!]}\cdot\left(\prod_{k=0}^{n}\binom{n+1}{k}\right)
\end{aligned}
$$

3. **Solution.**

   Denote by $P(n)$ the following proposition:

   $(1+x)^n = \binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\cdots+\binom{n}{k}x^k+\cdots+\binom{n}{n-1}x^{n-1}+\binom{n}{n}x^n$ as polynomials.

   - Note that $(1+x)^0 = 1 = \binom{0}{0}$ as polynomials. Hence $P(0)$ is true.
   - Let $m \in \mathbb{N}$. Suppose $P(m)$ is true. Then

     $$(1+x)^m = \binom{m}{0}+\binom{m}{1}x+\binom{m}{2}x^2+\cdots+\binom{m}{k}x^k+\cdots+\binom{m}{m-1}x^{m-1}+\binom{m}{m}x^m$$

     as polynomials.

     We verify that $P(m+1)$ is true:

As polynomials,

$$
\begin{aligned}
& (1+x)^{m+1} = (1+x)(1+x)^m \\
= \ & (1+x)\Big(\binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \cdots \\
& \quad + \binom{m}{k}x^k + \cdots + \binom{m}{m-1}x^{m-1} + \binom{m}{m}x^m\Big) \\
= \ & \Big(\binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \cdots + \binom{m}{k}x^k + \binom{m}{k+1}x^{k+1} + \cdots + \binom{m}{m}x^m\Big) \\
& + \Big(\binom{m}{0}x + \binom{m}{1}x^2 + \cdots \\
& \quad + \binom{m}{k-1}x^k + \binom{m}{k}x^{k+1} + \cdots + \binom{m}{m-1}x^m + \binom{m}{m}x^{m+1}\Big) \\
= \ & \binom{m}{0} + \Big(\binom{m}{0} + \binom{m}{1}\Big)x + \Big(\binom{m}{1} + \binom{m}{2}\Big)x^2 + \cdots + \Big(\binom{m}{k-1} + \binom{m}{k}\Big)x^k \\
& + \Big(\binom{m}{k} + \binom{m}{k+1}\Big)x^{k+1} + \cdots + \Big(\binom{m}{m-1} + \binom{m}{m}\Big)x^m + \binom{m}{m}x^{m+1} \\
= \ & \binom{m+1}{0} + \binom{m+1}{1}x + \binom{m+1}{2}x^2 + \cdots + \binom{m+1}{k}x^k + \binom{m+1}{k+1}x^{k+1} \\
& + \cdots + \binom{m+1}{m}x^m + \binom{m+1}{m+1}x^{m+1}
\end{aligned}
$$

It follows that $P(m+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

4. (a) **Solution.**

Let n be a positive integer, and $f(x)$ be the polynomial $f(x) = (1+x)^n$.

Note that $f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k$ as polynomials.

i. $\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^k = f(1) = (1+1)^n = 2^n$.

ii. $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = f(-1) = (1-1)^n = 0$.

iii. $\sum_{k=0}^{n} \frac{1}{2^k} \binom{n}{k} = f(\frac{1}{2}) = \left(1 + \frac{1}{2}\right)^n = \frac{3^n}{2^n}$.

iv. $\sum_{k=0}^{n} \frac{(-1)^k \cdot 3^{k-1}}{5^{k+1}} \binom{n}{k} = \frac{1}{15} \sum_{k=0}^{n} \frac{(-1)^k \cdot 3^k}{5^k} \binom{n}{k} = \frac{1}{15} f(-\frac{3}{5}) = \frac{1}{15}\left(1 - \frac{3}{5}\right)^n = \frac{2^n}{15 \cdot 5^n}$.

(b) **Solution.**

Let m be a positive integer. Then $2m$ is a positive integer.

Let $g(x)$ be the polynomial $g(x) = (1+x)^{2m}$.

Note that $g(x) = \sum_{k=0}^{2m} \binom{2m}{k} x^k$ as polynomials.

i. $\sum_{k=0}^{2m} \binom{2m}{k} = g(1) = 2^{2m}$.

ii. $\sum_{k=0}^{2m} (-1)^k \binom{2m}{k} = g(-1) = 0$.

iii.

$$
\begin{aligned}
\sum_{k=0}^{m} \binom{2m}{2k} &= \sum_{j=0}^{2m} \frac{1}{2}\left(\binom{2m}{j} + (-1)^j \binom{2m}{j}\right) \\
&= \frac{1}{2}\left(\sum_{j=0}^{2m} \binom{2m}{j} + \sum_{j=0}^{2m} (-1)^j \binom{2m}{j}\right) = \frac{1}{2}(2^{2m} + 0) = 2^{2m-1}
\end{aligned}
$$

iv.

$$\sum_{k=0}^{m-1}\binom{2m}{2k+1} = \sum_{j=0}^{2m}\frac{1}{2}(\binom{2m}{j} - (-1)^j\binom{2m}{j})$$
$$= \frac{1}{2}(\sum_{j=0}^{2m}\binom{2m}{j} - \sum_{j=0}^{2m}(-1)^j\binom{2m}{j}) = \frac{1}{2}(2^{2m}-0) = 2^{2m-1}$$

(c) *Hint.* Note that $4p$ is a positive integer. Define the polynomial $h(z) = (1+z)^{4p}$.
The results in the previous parts give

$$\sum_{k=0}^{4p}\binom{4p}{k} = 2^{4p}, \qquad \sum_{k=0}^{4p}(-1)^k\binom{4p}{k} = 0, \qquad \sum_{k=0}^{2p}\binom{4p}{2k} = 2^{4p-1}, \qquad \sum_{k=0}^{2p-1}\binom{4p}{2k+1} = 2^{4p-1}.$$

Further obtain

$$(-1)^p \cdot 2^{2p} = h(i) = \sum_{j=0}^{2p}(-1)^j\binom{4p}{2j} + i\sum_{j=0}^{2p-1}(-1)^j\binom{4p}{2j+1}$$

Make use of these above to the answers for the respective parts.
Answer.

i. $(-1)^p \cdot 2^{2p-1}$.
ii. 0.
iii. $2^{4p-2} + (-1)^p \cdot 2^{2p-1}$.
iv. $2^{4p-2} - (-1)^p \cdot 2^{2p-1}$.
v. 2^{4p-2}.
vi. 2^{4p-2}.

5. **Solution.**

(a) Let $n \in \mathbb{N}\backslash\{0\}$, and $k \in \mathbb{Z}$.

- (Case 1.) Suppose $0 < k \leq n$. Then

$$k \cdot \binom{n}{k} = k \cdot \frac{n!}{k! \cdot (n-k)!} = \frac{n!}{(k-1)! \cdot (n-k)!} = n \cdot \frac{(n-1)!}{(k-1)! \cdot [(n-1)-(k-1)]!} = n \cdot \binom{n-1}{k-1}.$$

- (Case 2.) Suppose $k \leq 0$ or $k > n$. Then $k \cdot \binom{n}{k} = 0 = n \cdot \binom{n-1}{k-1}$.

Hence in any case, $k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$.

(b) Let n be a positive integer.

i. $\sum_{k=0}^{n} k\binom{n}{k} = \sum_{k=1}^{n} k\binom{n}{k} = \sum_{k=1}^{n} n\binom{n-1}{k-1} = n\sum_{k=1}^{n}\binom{n-1}{k-1} = n\sum_{j=0}^{n-1}\binom{n-1}{j} = n \cdot 2^{n-1}$.

Alternative argument. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = (1+x)^n$ for any $x \in \mathbb{R}$.
f is smooth on $\mathbb{R}$, and for any $x \in \mathbb{R}$, we have

$$n(1+x)^{n-1} = f'(x) = \sum_{k=1}^{n} k\binom{n}{k}x^{k-1}.$$

Then $\sum_{k=0}^{n} k\binom{n}{k} = f'(1) = n \cdot 2^{n-1}$.

ii. We have $n \leq 1$ or $n \geq 2$.

- (Case 1.) Suppose $n = 1$. Then We have $\sum_{k=0}^{n}(-1)^{k+1}k\binom{n}{k} = \sum_{k=0}^{1}(-1)^{k+1}k\binom{n}{k} = (-1)^{1+1} \cdot 1 \cdot \binom{1}{1} = 1$.

- (Case 2.) Suppose $n \geq 2$. Then we have

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^{k+1}k\binom{n}{k} &= \sum_{k=1}^{n}(-1)^{k+1}k\binom{n}{k} \\
&= \sum_{k=1}^{n}(-1)^{k+1}n\binom{n-1}{k-1} \\
&= n\sum_{k=1}^{n}(-1)^{k+1}\binom{n-1}{k-1} \\
&= n\sum_{j=0}^{n-1}(-1)^{j+2}\binom{n-1}{j} \\
&= n\sum_{j=0}^{n-1}(-1)^{j}\binom{n-1}{j} = n\cdot 0 = 0
\end{aligned}$$

iii. We have $n \geq 2$ or $n \leq 1$.

- (Case 1.) Suppose $n \geq 2$. Then

$$\begin{aligned}
\sum_{k=0}^{n}k(k-1)\binom{n}{k} &= \sum_{k=2}^{n}k(k-1)\binom{n}{k} \\
&= \sum_{k=2}^{n}n(k-1)\binom{n-1}{k-1} \\
&= \sum_{k=2}^{n}n(n-1)\binom{n-2}{k-2} \\
&= n(n-1)\sum_{k=2}^{n}\binom{n-2}{k-2} \\
&= n(n-1)\sum_{j=0}^{n-2}\binom{n-2}{j} = n(n-1)\cdot 2^{n-2}
\end{aligned}$$

- (Case 2). Suppose $n \leq 1$. Then $\sum_{k=0}^{n}k(k-1)\binom{n}{k} = 0 = n(n-1)\cdot 2^{n-2}$.

Hence in any case, $\sum_{k=0}^{n}k(k-1)\binom{n}{k} = n(n-1)\cdot 2^{n-2}$.

iv.

$$\begin{aligned}
\sum_{k=0}^{n}k^2\binom{n}{k} &= \sum_{k=0}^{n}[k(k-1)+k]\binom{n}{k} \\
&= \sum_{k=0}^{n}k(k-1)\binom{n}{k} + \sum_{k=0}^{n}k\binom{n}{k} \\
&= n(n-1)\cdot 2^{n-2} + n\cdot 2^{n-1} = n(n+1)\cdot 2^{n-2}
\end{aligned}$$

(c) Let m be a positive integer. Note that $m+1 > 1$.

For each non-negative integer k, we have $(k+1)\cdot\binom{m+1}{k+1} = (m+1)\cdot\binom{m}{k}$.

Therefore

$$\frac{1}{k+1}\cdot\binom{m}{k} = \frac{1}{m+1}\cdot\binom{m+1}{k+1}.$$

i.

$$\begin{aligned}
\sum_{k=0}^{m} \frac{1}{k+1}\binom{m}{k} &= \sum_{k=0}^{m} \frac{1}{m+1}\binom{m+1}{k+1} \\
&= \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k+1} \\
&= \frac{1}{m+1}\sum_{j=1}^{m+1}\binom{m+1}{j} \\
&= \frac{1}{m+1}(\sum_{j=0}^{m+1}\binom{m+1}{j} - 1) = \frac{2^{m+1}-1}{m+1}
\end{aligned}$$

Alternative argument. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = (1+x)^n$ for any $x \in \mathbb{R}$.
f is continuous on $\mathbb{R}$, and for any $x \in \mathbb{R}$, we have

$$\frac{(1+x)^{n+1}-1}{n+1} = \int_0^x f(t)dt = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} x^{k+1}.$$

Then $\displaystyle\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \int_0^1 f(t)dt = \frac{2^{n+1}-1}{n+1}$.

ii.

$$\begin{aligned}
\sum_{k=0}^{m} \frac{(-1)^k}{k+1}\binom{m}{k} &= \sum_{k=0}^{m} \frac{(-1)^k}{m+1}\binom{m+1}{k+1} \\
&= \frac{1}{m+1}\sum_{k=0}^{m}(-1)^k\binom{m+1}{k+1} \\
&= \frac{1}{m+1}\sum_{j=1}^{m+1}(-1)^{j-1}\binom{m+1}{j} \\
&= \frac{1}{m+1}(-\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j} + 1) = \frac{1}{m+1}
\end{aligned}$$

iii.

$$\begin{aligned}
\sum_{k=0}^{m} \frac{1}{(k+2)(k+1)}\binom{m}{k} &= \sum_{k=0}^{m} \frac{1}{m+1}\cdot\frac{1}{k+2}\binom{m+1}{k+1} \\
&= \sum_{k=0}^{m} \frac{1}{(m+2)(m+1)}\binom{m+2}{k+2} \\
&= \frac{1}{(m+2)(m+1)}\sum_{k=0}^{m}\binom{m+2}{k+2} \\
&= \frac{1}{(m+2)(m+1)}\sum_{j=2}^{m+2}\binom{m+2}{j} \\
&= \frac{1}{(m+2)(m+1)}[\sum_{j=0}^{m+2}\binom{m+2}{j} - 1 - (m+2)] = \frac{2^{m+2}-m-3}{(m+2)(m+1)}
\end{aligned}$$

6. **Solution.**
Let α be a complex number. Suppose $0 < |\alpha| < 1$.

Define $\beta = \dfrac{1}{|\alpha|} - 1$. We have $\beta > 0$.

(a) Suppose $n \geq 2$. Then, by Bernoulli's Inequality, $(1+\beta)^n \geq 1 + n\beta \geq n\beta$. Therefore $|\alpha|^n = \dfrac{1}{(1+\beta)^n} \leq \dfrac{1}{n\beta}$.

(b) Suppose $n \geq 3$. Then $(1+\beta)^n = 1 + n\beta + \dfrac{n(n-1)}{2}\beta^2 + \underbrace{\cdots\ \cdots\ \cdots}_{\text{finitely many non-negative terms}} \geq \dfrac{n(n-1)}{2}\beta^2$.

Therefore $n|\alpha|^n = \dfrac{n}{(1+\beta)^n} \leq \dfrac{2}{(n-1)\beta^2} = \dfrac{2}{[n/2 + (n/2-1)]\beta^2} \leq \dfrac{2}{(n/2)\beta^2} = \dfrac{4}{n\beta^2}$.

(c) Suppose $n \geq 4$. Then

$$(1+\beta)^n = 1 + n\beta + \frac{n(n-1)}{2}\beta^2 + \frac{n(n-1)(n-2)}{6}\beta^3 + \underbrace{\cdots \quad \cdots \quad \cdots}_{\text{finitely many non-negative terms}} \geq \frac{n(n-1)(n-2)}{6}\beta^3.$$

Therefore

$$n^2|\alpha|^n = \frac{n^2}{(1+\beta)^n} \leq \frac{6n}{(n-1)(n-2)\beta^3} = \frac{6n}{[n/2+(n/2-1)][n/3+(2n/3-2)]\beta^3} \leq \frac{6n}{(n/2)(n/3)\beta^3} = \frac{36}{n\beta^3}.$$

(d) *Hint.* Generalize what you see in part (b) and part (c).

7. ——

8. **Answer.**

(a) ——

(b) i. $f'(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ x^{-2}e^{-1/x} & \text{if} \quad x > 0 \end{cases}$.

ii. ——

iii. ——

(c) i. *Hint.* Apply mathematical induction to the proposition $S(n)$ below:

- There exists some polynomial function P_n such that

$$f^{(n)}(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ P_n(1/x)e^{-1/x} & \text{if} \quad x > 0 \end{cases}.$$

ii. *Hint.* Apply mathematical induction to the proposition $T(n)$ below:

- f is n-times differentiable at 0 and $f^{(n)}(0) = 0$.

Be careful: that f is n-times differentiable at 0 for each specific n is something which needs being argued for. It should not be taken for granted, even though it is apparent by the result of the previous part that f is smooth at every point of $\mathbb{R}$ other than 0.

Remark. The Taylor series $T_{f,0}(x)$ of the function f about the point 0 is given by the expression $\sum_{n=0}^{\infty} 0 \cdot x^n$. But f is not the zero function; in fact, for any $\delta > 0$, we have $f(\delta/2) \neq 0$. Therefore f fails to be equal to the constant zero function on $(0, \delta)$,

9. **Solution.**

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function. Suppose that $f(x+y) = f(x) + f(y)$ for any $x, y \in \mathbb{R}$.

(a) We have $f(0) = f(0+0) = f(0) + f(0)$. Then $f(0) = 0$.

(b) Let $x \in \mathbb{R}$. Note that $-x \in \mathbb{R}$.
We have $f(x) + f(-x) = f(x + (-x)) = f(0) = 0$. Then $f(-x) = -f(x)$.

(c) Denote by $P(n)$ the proposition $f(n) = nf(1)$.

- We have $f(0) = 0 = 0 \cdot f(1)$. Then $P(1)$ is true.
- Let $k \in \mathbb{N}$. Suppose $P(k)$ is true. Then $f(k) = kf(1)$.
  We verify that $P(k+1)$ is true:
  We have $f(k+1) = f(k) + f(1) = kf(1) + f(1) = (k+1)f(1)$.
  Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

(d) Let $m \in \mathbb{Z}$.

- Suppose $m \geq 0$. Then $m \in \mathbb{N}$. We have $f(m) = mf(1)$.
- Suppose $m < 0$. Then $-m > 0$.
  We have $-f(m) = f(-m) = (-m) \cdot f(1) = -mf(1)$.
  Then $f(m) = mf(1)$.

Hence, in any case, we have $f(m) = mf(1)$.

(e) Let $r \in \mathbb{Q}$. There exist some $m, n \in \mathbb{Z}$ such that $n \neq 0$ and $m = nr$.
Now $mf(1) = f(m) = f(nr) = nf(r)$.
Then $f(r) = \frac{m}{n}f(1) = rf(1)$.

(f) Now further suppose that f is continuous on $\mathbb{R}$.
Let $u \in \mathbb{R}$. There exists some infinite sequence of rational numbers $\{s_n\}_{n=0}^{\infty}$ such that $\lim_{n\to\infty} s_n = u$.
For each $n \in \mathbb{N}$, we have $f(s_n) = f(1)s_n$.
Then, by continuity, we have $f(u) = f\left(\lim_{n\to\infty} s_n\right) = \lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} f(1)s_n = f(1)u$.