MATH1050 Exercise 5

1. We introduce this definition below:

   - *Let $n \in \mathbb{N}$. Define* $\binom{n}{k} = \begin{cases} \dfrac{n!}{k! \cdot (n-k)!} & \textit{if} \quad k \in [\![0, n]\!], \\ 0 & \textit{if} \quad k \in \mathbb{Z} \textit{ and } (k < 0 \textit{ or } k > n). \end{cases}$

     The number $\binom{n}{k}$ *is usually referred to as the* **binomial coefficients** *of n over k.*

   (a) Let $n \in \mathbb{N}$, and $k \in \mathbb{Z}$.

   i. Verify that $\binom{n}{k} = \binom{n}{n-k}$.

   ii. Verify that $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$.

   (b)$^\diamond$ Apply the results above and the Telescopic Method (or mathematical induction) to verify the statements below:

   i. $\displaystyle\sum_{j=0}^{m} \binom{n+j}{j} = \binom{n+m+1}{m}$ *for any $n, m \in \mathbb{N}$.*

   ii. $\displaystyle\sum_{j=0}^{m} \binom{n+j}{n} = \binom{n+m+1}{n+1}$ *for any $n, m \in \mathbb{N}$.*

2. Let n be a positive integer.

   (a) Suppose r is an integer amongst $0, 1, \cdots, n$. Prove that $\binom{n+1}{r+1} / \binom{n+1}{r} = \dfrac{n+1-r}{r+1}$.

   (b) Hence, or otherwise, deduce the equalities below:

   i. $\displaystyle\sum_{k=0}^{n} (k+1) \cdot \binom{n+1}{k+1} / \binom{n+1}{k} = \frac{An^2 + Bn + C}{2}$.

   ii. $\displaystyle\prod_{k=0}^{n} \left(\binom{n+1}{k+1} + \binom{n+1}{k} \right) = \frac{(n+D)^{n+E}}{[(n+F)!]} \cdot \left(\prod_{k=0}^{n} \binom{n+1}{k} \right)$.

   Here A, B, C, D, E, F are some positive integers whose respective values you have to determine explicitly.

3. Apply mathematical induction to prove the **Binomial Theorem** (formulated in terms of polynomials):

   - *Suppose $n \in \mathbb{N}$. Then* $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{k}x^k + \cdots + \binom{n}{n-1}x^{n-1} + x^n$ *as polynomials.*

4. (a) Let n be a positive integer. By considering the polynomial $(1+x)^n$, or otherwise, find the respective values of the numbers below. Leave your answer in terms of n where appropriate.

   i. $\displaystyle\sum_{k=0}^{n} \binom{n}{k}$.

   ii. $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k}$.

   iii. $\displaystyle\sum_{k=0}^{n} \frac{1}{2^k} \binom{n}{k}$.

   iv. $\displaystyle\sum_{k=0}^{n} \frac{(-1)^k \cdot 3^{k-1}}{5^{k+1}} \binom{n}{k}$.

   (b)$^\diamond$ Let m be a positive integer. By consider the polynomial $(1+x)^{2m}$, or otherwise, find the respective values of the numbers below. Leave your answer in terms of m where appropriate.

   i. $\displaystyle\sum_{k=0}^{2m} \binom{2m}{k}$.

   ii. $\displaystyle\sum_{k=0}^{2m} (-1)^k \binom{2m}{k}$.

   iii. $\displaystyle\sum_{k=0}^{m} \binom{2m}{2k}$.

   iv. $\displaystyle\sum_{k=0}^{m-1} \binom{2m}{2k+1}$.

   (c)$^\clubsuit$ Let p be a positive integer. By consider the polynomial $(1+x)^{4p}$, or otherwise, find the respective values of the numbers below. Leave your answer in terms of p where appropriate. (*Hint.* Make good use of complex numbers.)

   i. $\displaystyle\sum_{j=0}^{2p} (-1)^j \binom{4p}{2j}$

   ii. $\displaystyle\sum_{j=0}^{2p-1} (-1)^j \binom{4p}{2j+1}$

   iii. $\displaystyle\sum_{k=0}^{p} \binom{4p}{4k}$.

   iv. $\displaystyle\sum_{k=0}^{p-1} \binom{4p}{4k+2}$.

   v. $\displaystyle\sum_{k=0}^{p-1} \binom{4p}{4k+1}$.

   vi. $\displaystyle\sum_{k=0}^{p-1} \binom{4p}{4k+3}$.

5. (a) Let $n \in \mathbb{N} \backslash \{0\}$, and $k \in \mathbb{Z}$. Prove that $k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$.

   (b) Let n be a positive integer. Find the respective values of the numbers below. Leave your answer in terms of n where appropriate.

   i. $\displaystyle\sum_{k=0}^{n} k \binom{n}{k}$.

   ii.$^\diamond$ $\displaystyle\sum_{k=0}^{n} (-1)^{k+1} k \binom{n}{k}$.

   iii.$^\clubsuit$ $\displaystyle\sum_{k=0}^{n} k(k-1) \binom{n}{k}$.

   iv. $\displaystyle\sum_{k=0}^{n} k^2 \binom{n}{k}$.

   Remark. There is an alternative method for computing the sums described here: make use of differentiation.

   (c)$^\diamond$ Let m be a positive integer. Find the respective values of the numbers below. Leave your answer in terms of m where appropriate.

i. $\sum_{k=0}^{m} \frac{1}{k+1} \binom{m}{k}$. ii.$^\diamondsuit$ $\sum_{k=0}^{m} \frac{(-1)^k}{k+1} \binom{m}{k}$. iii.$^\clubsuit$ $\sum_{k=0}^{m} \frac{1}{(k+2)(k+1)} \binom{m}{k}$.

Remark. There is an alternative method for computing the sums described here: make use of integration.

6. Let α be a complex number. Suppose $0 < |\alpha| < 1$. Define the number β by $\beta = \frac{1}{|\alpha|} - 1$. Note that $\beta > 0$.

   Let n be a positive integer.

   (a) Suppose $n \geq 2$. By applying Bernoulli's Inequality, or the Binomial Theorem, prove that $|\alpha|^n \leq \frac{1}{n\beta}$.

   (b)$^\diamondsuit$ Suppose $n \geq 3$. By applying the Binomial Theorem, or otherwise, prove that $(1+\beta)^n \geq \frac{n(n-1)}{2}\beta^2$.

   Hence deduce that $n|\alpha|^n \leq \frac{4}{n\beta^2}$.

   (c)$^\clubsuit$ Suppose $n \geq 4$. By applying the Binomial Theorem prove that $(1+\beta)^n \geq \frac{n(n-1)(n-2)}{6}\beta^3$.

   Hence deduce that $n^2|\alpha|^n \leq \frac{36}{n\beta^3}$.

   (d)$^\heartsuit$ Let k be a non-negative integer. Suppose $n \geq k+2$.

   By applying the Binomial Theorem, or otherwise, prove that $n^k|\alpha|^n \leq \frac{[(k+1)!]^2}{n\beta^{k+1}}$.

   Remark. The inequalities described here constitute the key step in the argument for the statement '$\lim_{n\to\infty} n^k\alpha^n = 0$'.

7.$^\clubsuit$ *Familiarity with the calculus of one variable is assumed in this question.*

   Apply mathematical induction to prove **Leibniz's Rule (on repeated differentiation for products of functions)**:

   - *Suppose g, h are functions which are n-times differentiable at a. Then $g \cdot h$ is n-times differentiable at a and* $(g \cdot h)^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} g^{(k)}(a)h^{(n-k)}(a)$.

   Remark. Hence we have the generalization of the 'product rule' for higher-order derivatives. Note the role of binomial coefficients in the 'formula' in Leibniz's Rule. When giving the argument, you may take for granted the validity of the 'product rule' for computing first derivatives.

8. *Familiarity with the calculus of one variable is assumed in this question.*

   Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x} & \text{if } x > 0 \end{cases}$.

   Take for granted that $\lim_{x\to 0^+} e^{-1/x} = 0$. *Also take for granted that the function f is smooth at every point in $\mathbb{R}\backslash\{0\}$. (The point in this question is the behaviour of the function f at and near 0.)*

   (a) By applying L'Hôpital's Rule, or otherwise, verify that $\lim_{x\to 0^+} x^{-n}e^{-1/x} = 0$ for any positive integer n. *Take for granted the existence of the respective limits.*

   (b) i. Compute $f'(x)$ for each $x \in \mathbb{R}\backslash\{0\}$.
   ii. Verify that f is differentiable at 0.
   iii. Verify that f is continuously differentiable at 0.

   (c) i.$^\clubsuit$ Prove that for each positive integer n, there exists some polynomial function P_n such that

   $$f^{(n)}(x) = \begin{cases} 0 & \text{if } x < 0 \\ P_n(1/x)e^{-1/x} & \text{if } x > 0 \end{cases}$$

   ii. Prove that f is smooth at 0. Also find the value of $f^{(n)}(0)$ for each $n \in \mathbb{N}$.

9. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function. Suppose that $f(x+y) = f(x) + f(y)$ for any $x, y \in \mathbb{R}$.

   (a) Prove that $f(0) = 0$.
   (b) Prove that $f(-x) = -f(x)$ for any $x \in \mathbb{R}$.
   (c) Prove that $f(n) = nf(1)$ for any $n \in \mathbb{N}$.
   (d) Prove that $f(m) = mf(1)$ for any $m \in \mathbb{Z}$.
   (e) Prove that $f(r) = rf(1)$ for any $r \in \mathbb{Q}$.
   (f) *Familiarity with the calculus of one variable is assumed in this part.*
   Take for granted the validity of the result below:

   - *For any $u \in \mathbb{R}$, there exists some infinite sequence of rational numbers $\{s_n\}_{n=0}^{\infty}$ such that* $\lim_{n\to\infty} s_n = u$.

   Now further suppose that f is continuous on $\mathbb{R}$.
   Prove that there exists some $c \in \mathbb{R}$ such that $f(x) = cx$ for any $x \in \mathbb{R}$.