MATH1050 Exercise 4 Supplement

1. Apply mathematical induction to justify each of the statements below:

   (a) $\dfrac{1}{1\cdot 4}+\dfrac{1}{4\cdot 7}+\dfrac{1}{7\cdot 10}+\cdots+\dfrac{1}{(3n-2)(3n+1)}=\dfrac{n}{3n+1}$ *for any* $n\in\mathbb{N}\backslash\{0\}$.

   (b) $-1^2+2^2-3^2+4^2+\cdots+(-1)^n n^2=(-1)^n\cdot\dfrac{n(n+1)}{2}$ *whenever* $n$ *is a positive integer.*

   (c) $\left(1-\dfrac{1}{2^2}\right)\left(1-\dfrac{1}{3^2}\right)\left(1-\dfrac{1}{4^2}\right)\cdot\ldots\cdot\left(1-\dfrac{1}{n^2}\right)=\dfrac{n+1}{2n}$ *for any* $n\in\mathbb{N}\backslash\{0,1\}$.

   (d) $\dfrac{0}{2^0}+\dfrac{1}{2^1}+\dfrac{2}{2^2}+\cdots+\dfrac{n}{2^n}=2-\dfrac{n+2}{2^n}$ *for any* $n\in\mathbb{N}$.

   (e) $(1^2+1)\cdot(1!)+(2^2+2)\cdot(2!)+(3^2+3)\cdot(3!)+\cdots(n^2+1)\cdot(n!)=n\cdot[(n+1)!]$ *for any* $n\in\mathbb{N}\backslash\{0\}$.

   (f)$^\diamondsuit$ $0^2+1^2+2^2+\cdots+(2n-1)^2=\dfrac{n(2n-1)(4n-1)}{3}$ *for each positive integer* $n$.

   (g)$^\diamondsuit$ $\displaystyle\sum_{k=0}^{n}k^5=\dfrac{n^2(n+1)^2(2n^2+2n-1)}{12}$ *for any* $n\in\mathbb{N}$.

   (h)$^\clubsuit$ $\displaystyle\sum_{k=0}^{n}k^4=\dfrac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$ *for any* $n\in\mathbb{N}$.

   (i)$^\diamondsuit$ $\displaystyle\sum_{k=1}^{2n}\dfrac{1}{k(k+1)}=\dfrac{2n}{2n+1}$ *for any positive integer* $n$.

2. Apply mathematical induction to prove the statements below. You have to think carefully which proposition is to be formulated and proved by mathematical induction.

   (a) *Suppose* $\alpha$ *is a number, not equal to* 1. *Then* $\displaystyle\sum_{k=1}^{n}\alpha^{k-1}=\dfrac{1-\alpha^n}{1-\alpha}$ *for each positive integer* $n$.

   (b)$^\diamondsuit$ *Suppose* $\alpha$ *is a number, not equal to* 1. *Then* $\displaystyle\sum_{k=1}^{n}k\alpha^{k-1}=\dfrac{1-(n+1)\alpha^n+n\alpha^{n+1}}{(1-\alpha)^2}$ *for each positive integer* $n$.

3. Apply mathematical induction to justify each of the statements below:

   (a) $n^2<2^n$ *whenever* $n$ *is an integer greater than* 4.

   (b) $n^3<3^n$ *for any integer greater than* 4.

   (c) $\dfrac{1}{1^2}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\cdots+\dfrac{1}{n^2}\le 2-\dfrac{1}{n}$ *for any positive integer* $n$.

   (d) $\dfrac{n}{2}<\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{2^n}<n$ *for any* $n\in\mathbb{N}\backslash\{0,1\}$.

   (e) $\displaystyle\prod_{k=1}^{n}[(2k)!]>[(n+1)!]^n$ *whenever* $n$ *is an integer greater than* 1.

4. Apply mathematical induction to justify each of the statements below:

   (a) $n(2n^2+1)$ *is divisible by* 3 *for any* $n\in\mathbb{N}$.

   (b) $(2n+1)(2n+3)(2n+5)$ *is divisible by* 3 *for any* $n\in\mathbb{N}$.

   (c) $(2n+1)(2n+3)(2n+5)(2n+7)(2n+9)$ *is divisible by* 5 *for any* $n\in\mathbb{N}$.

   (d) $2^{4n+3}+3^{3n+1}$ *is divisible by* 11 *for any* $n\in\mathbb{N}$.

   (e) $2^{n+1}+3^{2n-1}$ *is divisible by* 7 *for any positve integer* $n$.

5.$^\diamondsuit$ Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of numbers. Let $\alpha,\beta$ be numbers, with $\alpha\neq 1$. Suppose $a_{n+1}=\alpha a_n+\beta$ for each $n\in\mathbb{N}$.

   Apply the Telescopic Method to prove that $a_n=\alpha^n a_0+\dfrac{\beta(1-\alpha^n)}{1-\alpha}$ for each $n\geq 1$.

6. Justify the statements below, by applying the Telescopic Method, or by mathematical induction. If you choose mathematical induction, you have to think carefully which proposition is to be formulated and proved by mathematical induction.

   (a) *Let* $\theta \in \mathbb{R}$. *Suppose* $\sin(\theta) \neq 0$. *Then* $\cos(\theta)\cos(2\theta)\cos(2^2\theta)\cdot \ldots \cdot \cos(2^n\theta) = \dfrac{\sin(2^{n+1}\theta)}{2^{n+1}\sin(\theta)}$ *for any* $n \in \mathbb{N}$.

   (b) *Let* $\theta \in \mathbb{R}$. *Suppose* $\sin(\frac{\theta}{2}) \neq 0$. *Then* $1 + 2\sum_{k=1}^{n}\cos(k\theta) = \dfrac{\sin((n+1/2)\theta)}{\sin(\theta/2)}$ *for any* $n \in \mathbb{N}$.

   (c) *Let* $\theta \in \mathbb{R}$. *Suppose* $\sin(2^p\theta) \neq 0$ *for any* $p \in \mathbb{N}$. *Then* $\sum_{k=0}^{n} 2^k \tan(2^k\theta) = \cot(\theta) - 2^{n+1}\cot(2^{n+1}\theta)$ *for any* $n \in \mathbb{N}$.

   (d) *Let* $\theta \in \mathbb{R}$. *Suppose* $\sin(2^p\theta) \neq 0$ *for any* $p \in \mathbb{N}$. *Then* $\sum_{k=1}^{n} \csc(2^k\theta) = \cot(\theta) - \cot(2^n\theta)$ *for any* $n \in \mathbb{N}\backslash\{0\}$.

7. (a) Suppose $\theta, \alpha$ are real numbers.

      i. Verify that $\cos(\theta + k\alpha)\sin(\alpha) = \dfrac{1}{2}\left(\sin(\theta + \dfrac{2k+1}{2}\alpha) - \sin(\theta + \dfrac{2k-1}{2}\alpha)\right)$ for each integer $k$.

      ii.$^\diamond$ Now suppose $\sin(\frac{\alpha}{2}) \neq 0$ also. By applying mathematical induction, or otherwise, prove that

      $$\sum_{k=0}^{n}\cos(\theta + k\alpha) = \frac{\cos(\theta + n\alpha/2)\sin((n+1)\alpha/2)}{\sin(\alpha/2)} \text{ for each } n \in \mathbb{N}.$$

   (b)$^\clubsuit$ Suppose $\sin(\beta) \neq 0$. By applying the results above, or otherwise, prove that

   $$\sum_{k=1}^{2m}\cos^2(k\beta) = \frac{\cos(Am\beta)\sin((Bm+C)\beta)}{D\sin(\beta)} + \frac{Em+F}{2} \text{ and } \sum_{k=1}^{2m}\sin^2(k\beta) = \frac{\cos(Am\beta)\sin((Bm+C)\beta)}{D\sin(\beta)} + \frac{Gm+H}{2}$$

   for each positive integer $m$.

   Here $A, B, C, D, E, F, G, H$ are integers whose respective values you have to determine explicitly.

8. Apply mathematical induction to prove the statement below:

   - *Let* $\{a_n\}_{n=1}^{\infty}$ *be the infinite sequence of real numbers defined by*

   $$\begin{cases} a_1 &= 0 \\ a_{n+1} &= 2n - a_n \quad \text{if} \quad n \geq 1 \end{cases}.$$

   *Then* $a_n = n + \dfrac{(-1)^n - 1}{2}$ *for each positive integer* $n$.

9. Apply mathematical induction to prove the statement below:

   - *Let* $a, b$ *be distinct positive real numbers, and* $\{c_n\}_{n=1}^{\infty}$ *be the infinite sequence of real numbers defined by*

   $$\begin{cases} c_1 &= a + b \\ c_{n+1} &= a + b - \dfrac{ab}{c_n} \quad \text{if} \quad n \geq 1 \end{cases}.$$

   *Then* $c_n = \dfrac{a^{n+1} - b^{n+1}}{a^n - b^n}$ *for each positive integer* $n$.

10.$^\clubsuit$ Prove the statement below:

   - *Let* $\alpha, \beta$ *are the two distinct roots of the polynomial* $f(x) = x^2 - 2x - 1$. *Let* $\{a_n\}_{n=1}^{\infty}$ *be the infinite sequence of real numbers defined by*

   $$\begin{cases} a_1 = 1, & a_2 = 3, \\ & a_{n+2} = 2a_{n+1} + a_n \quad \text{if} \quad n \geq 1 \end{cases}.$$

   *Then* $a_n = \dfrac{1}{2}(\alpha^n + \beta^n)$ *for each positive integer* $n$.

   **Remark.** You have to think carefully which proposition is to be formulated and proved by mathematical induction.

11.♣ Apply mathematical induction to prove the statement below:

- *Let $\{a_n\}_{n=0}^{\infty}$ be the infinite sequence of real numbers defined by*

$$\begin{cases} a_0 = 1, \quad a_1 = 6, \quad a_2 = 45, \\ a_{n+3} = 9a_{n+2} - 27a_{n+1} + 27a_n \quad \text{if} \quad n \geq 0 \end{cases}.$$

*Then $a_n = 3^n(n^2+1)$ for each $n \in \mathbb{N}$.*

**Remark.** You have to think carefully which proposition is to be formulated and proved by mathematical induction.

12.♣ Apply mathematical induction to prove the statement below:

- *Let $\{a_n\}_{n=1}^{\infty}$ be an infinite sequence in $\mathbb{N}$. Suppose $n \leq \sum_{j=1}^{n} {a_j}^2 \leq n+1+(-1)^n$ for each positive integer $n$. Then $a_n = 1$ for each positive integer $n$.*

**Remark.** You have to think carefully which proposition is to be formulated and proved by mathematical induction.

13.♣ Apply mathematical induction to justify each of the statements below. You have to think carefully which proposition is to be formulated and proved by mathematical induction.

(a) For any $n \in \mathbb{N}$, $(\sqrt{3}+1)^{2n+1} - (\sqrt{3}-1)^{2n+1}$ is an integer which is divisible by $2^{n+1}$.

(b) For any $n \in \mathbb{N}$, $(3+\sqrt{5})^{n+1} + (3-\sqrt{5})^{n+1}$ is an integer which is divisible by $2^{n+1}$.

14. Prove the statement below:

- *Suppose $a, b$ are positive real numbers. Then $\dfrac{a^n+b^n}{2} \geq \left(\dfrac{a+b}{2}\right)^n$ for any $n \in \mathbb{N}\backslash\{0\}$.*

15. (a) Let $u, v, x, y$ be real numbers, and $\zeta = u+vi$, $\eta = x+yi$.
By considering the number $\zeta\bar{\eta}$, or otherwise, deduce the inequality $(ux+vy)^2 \leq (u^2+v^2)(x^2+y^2)$.

(b)◇ Apply mathematical induction, with the help of the result above where appropriate, to prove that the statement below:

- *Let $a, b, c$ be positive real numbers. Suppose $a^2+b^2=c^2$. Then $a^n+b^n < c^n$ for each integer $n \geq 3$.*

16. Apply mathematical induction to prove the statement below:

- *Suppose $x$ is a positive real number. Then $\dfrac{x^n}{1+x+x^2+\cdots+x^{2n}} \leq \dfrac{1}{2n+1}$ for any positive integer $n$.*

17. (a) Apply mathematical induction to prove the statement below:

- *Suppose $x$ is a positive real number. Then $n(x^{2n+1}+1)+x^{2n+1}+x^{2n+2} \leq (n+1)(x^{2n+3}+1)$ for any $n \in \mathbb{N}$.*

(b) Hence, or otherwise, prove the statement below:

- *For any $a > 0$, for any $n \in \mathbb{N}\backslash\{0\}$, $a+a^2+a^3+\cdots+a^{2n} \leq n(a^{2n+1}+1)$.*

18. Apply mathematical induction to justify each of the statements below:

(a)◇ *Let $n \in \mathbb{N} \in \backslash\{0,1\}$. Suppose $z_1, z_2, \cdots, z_n$ are complex numbers. Then*

$$\sqrt{|z_1|^2+|z_2|^2+\cdots+|z_n|^2} \leq |z_1|+|z_2|+\cdots+|z_n|.$$

(b)◇ *Let $n \in \mathbb{N}\backslash\{0,1\}$. Suppose $\theta_1, \theta_2, \cdots, \theta_n \in (0,\pi)$. Then $|\sin(\theta_1+\theta_2+\cdots+\theta_n)| < \sin(\theta_1)+\sin(\theta_2)+\cdots+\sin(\theta_n)$.*

(c)♣ *Let $n \in \mathbb{N}\backslash\{0\}$. Suppose $a_1, a_2, \cdots, a_n$ are positive real numbers. Then*

$$(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right) \geq n^2.$$

(d)♣ *Let $s, t \in \mathbb{Q}$, with $t > 0$, and $n \in \mathbb{N}\backslash\{0\}$. There exist $a, b \in \mathbb{Q}$ such that $(s + \sqrt{t})^n = a + b\sqrt{t}$.*

**Remark.** You have to think carefully which proposition is to be formulated and proved by mathematical induction. Do you 'fix' $s, t$ right at the beginning, so that your proposition handles the same $s, t$ throughout the argument? Or do you accommodate all possible '$s, t$' inside the same proposition?

19. (a)◇ Apply mathematical induction to prove the statement below:

- *Let $n \in \mathbb{N}\backslash\{0, 1\}$. Suppose $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are real numbers. Further suppose $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$. Then*

$$n\sum_{j=1}^{n} a_j b_j - \left(\sum_{j=1}^{n} a_j\right)\left(\sum_{k=1}^{n} b_k\right) = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}(a_j - a_k)(b_j - b_k).$$

(b) Hence, or otherwise, prove the statement below, known as **Chebychev's Inequality**:

- *Let $n \in \mathbb{N}\backslash\{0, 1\}$. Suppose $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are real numbers. Further suppose $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$. Then*

$$\left(\frac{1}{n}\sum_{j=1}^{n} a_j\right)\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right) \leq \frac{1}{n}\sum_{j=1}^{n} a_j b_j$$

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20. (a)◇ Prove the statement below:

- *Suppose $\zeta, \eta$ are complex numbers, and $c$ is a positive real number. Then $|\zeta + \eta|^2 \leq (1 + c)|\zeta|^2 + (1 + \frac{1}{c})|\eta|^2$.*

(b)♣ Apply mathematical induction, together with the result above, to prove the statement below:

- *Let $n \in \mathbb{N}\backslash\{0, 1\}$. Suppose $z_1, z_2, \cdots, z_n$ are complex numbers, and $a_1, a_2, \cdots, a_n$ are positive real numbers. Further suppose $\displaystyle\sum_{j=1}^{n}\frac{1}{a_j} = 1$. Then $\left|\displaystyle\sum_{j=1}^{n} z_j\right|^2 \leq \displaystyle\sum_{j=1}^{n} a_j|z_j|^2$.*

21. (a) Let $f(x)$ be the quartic polynomial given by $f(x) = x^4 + x^3 + x^2 + x + 1$.
Prove that $f(x)$ has no real root.
(*Hint.* Apply the proof-by-contradication argument. Can you express $f(x)$ as the sum of a positive real number together with one or several whole squares involving the indeterminate $x$?)

(b) Let $a_0, a_1, a_2, \cdots, a_9$ form a geometric progression, with common ratio $r$. Suppose $a_0 \neq 0$. Further suppose $a_0 + a_1 + \cdots + a_9 = 244(a_0 + a_1 + a_2 + a_3 + a_4)$.

i. Prove that $(r^5 - M)(N + r + r^2 + r^3 + r^4) = 0$. Here $M, N$ are positive integers whose respective value you have to determine explicitly.

ii. Find the value of $r$.

22. (a)◇ Prove the statement below:

- *Let $p, q$ be distinct positive prime numbers. $\sqrt{pq}$ is irrational.*

**Remark.** You may need apply Euclid's Lemma for several times.

(b) i. Prove the statement below:

- *Let $a, b, c$ be rational numbers. Suppose $a, c$ are positive and $\sqrt{a}, \sqrt{c}$ are irrational numbers. Further suppose $\sqrt{a} = b + \sqrt{c}$. Then $b = 0$.*

ii. Hence, or otherwise, prove the statement below:

- *Let $s, t, u, v$ be rational numbers. Suppose $t, v$ are positive and $\sqrt{t}, \sqrt{v}$ are irrational numbers. Further suppose $s + \sqrt{t} = u + \sqrt{v}$. Then $s = u$ and $t = v$.*

(c) Let $A, B, p, q$ be positive integers. Suppose $\sqrt{B}$ is an irrational number. Further suppose $p, q$ are distinct prime numbers. Prove the statements below

i.◇ $\sqrt{A + 2\sqrt{B}} = \sqrt{p} + \sqrt{q}$ iff ($A = p + q$, $B = pq$, and $A > 2\sqrt{B}$).

ii.♣ $\sqrt{A + 2\sqrt{B}} = \sqrt{p} + \sqrt{q}$ iff $\sqrt{|A - 2\sqrt{B}|} = |\sqrt{p} - \sqrt{q}|$.