MATH1050 Exercise 4 (Answers and solution)

1. **Solution.**

(a) Denote by $P(n)$ the proposition

$$1 \cdot 2 + 2 \cdot 5 + 3 \cdot 8 + \cdots + n(3n-1) = n^2(n+1).$$

- Note that $1 \cdot 2 = 2 = 1^2(1+1)$. Then $P(1)$ is true.
- Let k be a positive integer. Suppose $P(k)$ is true. Then

$$1 \cdot 2 + 2 \cdot 5 + 3 \cdot 8 + \cdots + k(3k-1) = k^2(k+1).$$

We verify that $P(k+1)$ is true:

We have

$$\begin{aligned} & 1 \cdot 2 + 2 \cdot 5 + 3 \cdot 8 + \cdots + k(3k-1) + (k+1)[3(k+1)-1] \\ = \;& k^2(k+1) + (k+1)(3k+2) = (k+1)[k^2 + (3k+2)] = \cdots = (k+1)^2[(k+1)+1] \end{aligned}$$

Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any positive integer n.

(b) Denote by $P(n)$ the proposition

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}.$$

- We have $1 \geq \sqrt{1}$. Hence $P(1)$ is true.
- Let k be a positive integer. Suppose $P(k)$ is true. Then $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} \geq \sqrt{k}$.

We verify that $P(k+1)$ is true:

$$\begin{aligned} 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} &\geq \sqrt{k} + \frac{1}{\sqrt{k+1}} \\ &= \frac{\sqrt{k} \cdot \sqrt{k+1} + 1}{\sqrt{k+1}} \\ &\geq \frac{\sqrt{k} \cdot \sqrt{k} + 1}{\sqrt{k+1}} \\ &= \sqrt{k+1} \end{aligned}$$

Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true whenever n is a positive integer.

(c) Denote by $P(n)$ the proposition

$$n^2 < 2^{n-1}.$$

- We have $7^2 = 49 < 64 = 27 - 1$. Then $P(7)$ is true.
- Let k be an integer greater than 6. Suppose $P(k)$ is true. Then $k^2 < 2^{k-1}$. Therefore $2^{k-1} > 2^k$.

We have

$$\begin{aligned} 2^{(k+1)-1} - (k+1)^2 &= 2^k - (k^2 + 2k + 1) \\ &\geq 2^k - k^2 - 2k - k \\ &= 2^k - k^2 - 3k \\ &\geq 2^k k^2 - k \cdot k = 2(2^{k-1} - k^2) >\geq 0. \end{aligned}$$

Then $(k+1)^2 < 2^{(k+1)-1}$. Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true whenever k is an integer greater than 6.

(d) Denote by $P(n)$ the proposition that $n(n^2+2)$ is divisible by 3.

- We have $0 \cdot (0^2 + 2) = 0 = 3 \cdot 0$ and $0 \in \mathbb{Z}$. Hence $0 \cdot (0^2+2)$ is divisible by 3.
Then $P(0)$ is true.
- Let k be a positive integer. Suppose $P(k)$ is true. Then $k(k^2+2)$ is divisible by 3. Therefore there exists some $q \in \mathbb{Z}$ such that $k(k^2+2) = 3q$.
We verify that $P(k+1)$ is true:
We have

$$(k+1)[(k+1)^2+2] = k^3 + 3k^2 + 5k + 3 = k(k^2+2) + 3k^2 + 3 = 3q + 3k^2 + 3 = 3(q + k^2 + 1).$$

Note that $q + k^2 + 1 \in \mathbb{Z}$. Then $(k+1)[(k+1)^2+2]$ is divisible by 3.
Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

(e) Denote by $P(n)$ the proposition that $7^n(3n+1)-1$ is divisible by 9.

- We have $7^0(3 \cdot 0+1)-1=0$. 0 is divisible by 9. Then $P(0)$ is true.
- Let $k \in \mathbb{N}$. Suppose $P(k)$ is true. Then $7^k(3k+1)-1$ is divisible by 9. Therefore there exists some $q \in \mathbb{Z}$ such that $7^k(3k+1)-1=9q$.
  We verify that $P(k+1)$ is true:

$$\begin{aligned} 7^{k+1}[3(k+1)+1]-1 &= 7 \cdot 7^k[(3k+1)+3]-1 \\ &= 7 \cdot 7^k(3k+1)+3 \cdot 7^{k+1}-1 \\ &= 7 \cdot [7^k(3k+1)-1]+3(7^{k+1}-1)+9 \\ &= 7 \cdot 9q+3(7-1)\sum_{j=0}^{k} 7^j+9 = 9\left(7q+2\sum_{j=0}^{k} 7^j+1\right) \end{aligned}$$

  Since $q \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have $7q+2\sum_{j=0}^{k} 7^j+1 \in \mathbb{Z}$. Therefore $7^{k+1}[3(k+1)+1]-1$ is divisible by 9.
  Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

2. **Solution.**
Suppose $\{a_n\}_{n=0}^{\infty}$ is an infinite sequence of complex numbers. Apply mathematical induction to prove the statements below:

(a) Denote by $P(n)$ the proposition

$$\sum_{k=0}^{n}(a_{k+1}-a_k) = a_{n+1}-a_0.$$

- We have $\sum_{k=0}^{0}(a_{k+1}-a_k) = a_1-a_0 = a_{0+1}-a_0$.
  Therefore $P(0)$ is true.
- Let $m \in \mathbb{N}$. Suppose $P(m)$ is true. Then $\sum_{k=0}^{m}(a_{k+1}-a_k) = a_{m+1}-a_0$.
  We verify $P(m+1)$:
  We have

$$\begin{aligned} \sum_{k=0}^{m+1}(a_{k+1}-a_k) &= \sum_{k=0}^{m}(a_{k+1}-a_k)+(a_{m+2}-a_{m+1}) \\ &= (a_{m+1}-a_0)+(a_{m+2}-a_{m+1}) = a_{m+2}-a_0 = a_{(m+1)+1}-a_0. \end{aligned}$$

  Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

(b) Further suppose $a_j \neq 0$ for each $j \in \mathbb{N}$.
Denote by $P(n)$ the proposition

$$\prod_{k=0}^{n}(a_{k+1}-a_k) = \frac{a_{n+1}}{a_0}.$$

- We have $\prod_{k=0}^{0}(a_{k+1}-a_k) = \frac{a_1}{a_0} = \frac{a_{0+1}}{a_0}$.
  Therefore $P(0)$ is true.
- Let $m \in \mathbb{N}$. Suppose $P(m)$ is true. Then $\prod_{k=0}^{m}(a_{k+1}-a_k) = \frac{a_{m+1}}{a_0}$.
  We verify $P(m+1)$:
  We have

$$\prod_{k=0}^{m+1}\frac{a_{k+1}}{a_k} = \left(\sum_{k=0}^{m}\frac{a_{k+1}}{a_k}\right) \cdot \frac{a_{m+2}}{a_{m+1}} = \frac{a_{m+1}}{a_0} \cdot \frac{a_{m+2}}{a_{m+1}} = \frac{a_{m+2}}{a_0} = \frac{a_{(m+1)+1}}{a_0}.$$

  Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

3. **Solution.**

(a) Denote by $P(n)$ the proposition $\displaystyle\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \sum_{k=n+1}^{2n} \frac{1}{k}$.

- We have $\displaystyle\sum_{k=1}^{2\cdot 1} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} = \frac{1}{2} = \sum_{k=1+1}^{2\cdot 1} \frac{1}{k}$.
  Hence $P(1)$ is true.
- Let m be a positive integer. Suppose $P(m)$ is true. We deduce that $P(m+1)$ is true:

$$\begin{aligned}
&\sum_{k=1}^{2(m+1)} \frac{(-1)^{k+1}}{k} - \sum_{k=(m+1)+1}^{2(m+1)} \frac{1}{k} \\
= &\sum_{k=1}^{2m} \frac{(-1)^{k+1}}{k} + \frac{(-1)^{2m+1+1}}{2m+1} + \frac{(-1)^{2m+2+1}}{2m+2} - \sum_{k=m+1}^{2m} \frac{1}{k} + \frac{1}{m+1} - \frac{1}{2m+1} - \frac{1}{2m+2} \\
= &\cdots = 0
\end{aligned}$$

  Then $\displaystyle\sum_{k=1}^{2m} \frac{(-1)^{k+1}}{k} = \sum_{k=m+1}^{2m} \frac{1}{k}$.
  Hence $P(m+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any positive integer n.

(b) i. Let x be a real number. Suppose $x > 1$.
We have $\displaystyle\frac{1}{x} = \int_{x-1}^{x} \frac{dt}{x} < \int_{x-1}^{x} \frac{dt}{t} = \ln(x) - \ln(x-1) = \ln\left(\frac{x}{x-1}\right)$.
We also have $\displaystyle\ln\left(\frac{x+1}{x}\right) = \ln(x+1) - \ln(x) = \int_{x}^{x+1} \frac{dt}{t} < \int_{x}^{x+1} \frac{dt}{x} = \frac{1}{x}$
Therefore $\displaystyle\ln\left(\frac{x+1}{x}\right) < \frac{1}{x} < \ln\left(\frac{x}{x-1}\right)$.

ii. Let n be a positive integer.
For each $k = n+1, n+2, \cdots, 2n$, we have $\displaystyle\ln\left(\frac{k+1}{k}\right) < \frac{1}{k} < \ln\left(\frac{k}{k-1}\right)$.
Then $\displaystyle\sum_{k=n+1}^{2n} \frac{1}{k} > \sum_{k=n+1}^{2n} \ln\left(\frac{k+1}{k}\right) = \ln\left(\prod_{k=n+1}^{2n} \frac{k+1}{k}\right) = \ln\left(\frac{2n+1}{n+1}\right)$.
Also $\displaystyle\sum_{k=n+1}^{2n} \frac{1}{k} < \sum_{k=n+1}^{2n} \ln\left(\frac{k}{k-1}\right) = \ln\left(\prod_{k=n+1}^{2n} \frac{k}{k-1}\right) = \ln(2)$
Therefore $\displaystyle\ln\left(\frac{2n+1}{n+1}\right) < \sum_{k=n+1}^{2n} \frac{1}{k} < \ln(2)$.

iii. Let n be a positive integer. By the results in the previous parts, we have $\displaystyle\ln\left(2 - \frac{1}{n+1}\right) = \ln\left(\frac{2n+1}{n+1}\right) < \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} < \ln(2)$.
By the continuity of the logarithmic function, $\displaystyle\lim_{n\to\infty} \ln\left(2 - \frac{1}{n+1}\right) = \ln(2)$. Also $\displaystyle\lim_{n\to\infty} \ln(2) = \ln(2)$.
By the Sandwich Rule, the limit $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k}$ exists and is equal to $\ln(2)$.

4. **Answer.**

(I) Suppose $\displaystyle\sum_{j=0}^{n} a_j = \left(\frac{1+a_n}{2}\right)^2$ for each $n \in \mathbb{N}$.

(II) $a_n = 2n+1$.

(III) We have $\displaystyle a_0 = \sum_{j=0}^{0} a_j = \left(\frac{1+a_0}{2}\right)^2 = \frac{1}{4}(1 + 2a_0 + {a_0}^2)$. Then $(a_0 - 1)^2 = {a_0}^2 - 2a_0 + 1 = 0$. Therefore $a_0 = 1 = 2 \cdot 0 + 1$.

(IV) Let $k \in \mathbb{N}$. Suppose $P(k)$ is true.

(V) We have

$$\left(\frac{1+a_{k+1}}{2}\right)^2 = \sum_{j=0}^{k+1} a_j = \sum_{j=0}^{k} a_j + a_{k+1} = \left(\frac{1+a_k}{2}\right)^2 + a_{k+1} = \left[\frac{1+(2k+1)}{2}\right]^2 + a_{k+1} = (k+1)^2 + a_{k+1}.$$

Then $\frac{1}{4}(1+2a_{k+1}+{a_{k+1}}^2) = (k+1)^2 + a_{k+1}$.
Therefore $(a_{k+1}-1)^2 = {a_{k+1}}^2 - 2a_{k+1} + 1 = (2k+2)^2$.
Hence $a_{k+1} = 2k+3$ or $a_{k+1} = -2k-1$. Since $a_{k+1} > 0$, we have $a_{k+1} = 2k+3 = 2(k+1)+1$.

(VI) By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

5. **Answer.**

(I) $1 = -(-1) = \alpha + \beta$

(II) $3 = [-(-1)]^2 - 2(-1) = (\alpha+\beta)^2 - 2\alpha\beta = \alpha^2 + \beta^2$

(III) Let k be a positive integer. Suppose $P(k)$ is true.

(IV) $\alpha^{k+1} + \beta^{k+1}$

(V) $P(k)$

(VI) $a_{k+2} = a_{k+1} + a_k = (\alpha^{k+1}+\beta^{k+1}) + (\alpha^k + \beta^k) = \alpha^k(\alpha+1) + \beta^k(\beta+1) = \alpha^k \cdot \alpha^2 + \beta^k \cdot \beta^2 = \alpha^{k+2} + \beta^{k+2}$.

(VII) By the Principle of Mathematical Induction, $P(n)$ is true for each positive integer n.

6. **Solution.**

Suppose $a \in (-1, +\infty)$. Denote by $P(n)$ the proposition $(1+a)^n \geq 1 + na$.

- We have $(1+a)^2 = 1 + 2a + a^2 \geq 1 + 2 \cdot a$.
  Hence $P(2)$ is true.
- Let $k \in \mathbb{N}\backslash\{0,1\}$. Suppose $P(k)$ is true. Then $(1+a)^k \geq 1 + ka$.
  We verify that $P(k+1)$ is true:

$$\begin{aligned}(1+a)^{k+1} &= (1+a)^k(1+a) \\ &\geq (1+ka)(1+a) \quad \text{because } 1+a \geq 0, \\ &= 1 + (k+1)a + ka^2 \\ &\geq 1 + (k+1)a \quad \text{because } ka^2 \geq 0.\end{aligned}$$

  Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}\backslash\{0,1\}$.

7. **Solution.**

(a) (I) Suppose $\mu_1, \cdots, \mu_n \in \mathbb{C}$. Then $\left|\sum_{j=1}^{n} \mu_j\right| \leq \sum_{j=1}^{n} |\mu_j|$.

(II) Let μ_1, μ_2 be complex numbers.
$\left|\sum_{j=1}^{2} \mu_j\right| = |\mu_1 + \mu_2| \leq |\mu_1| + |\mu_2| = \sum_{j=1}^{2} |\mu_j|$.

(III) Let $k \in \mathbb{N}\backslash\{0,1\}$. Suppose $P(k)$ is true.

(IV) Let $\nu_1, \cdots, \nu_k, \nu_{k+1}$ be complex numbers. We have

$$\left|\sum_{j=1}^{k+1} \nu_j\right| = \left|\sum_{j=1}^{k} \nu_j + \nu_{k+1}\right| \leq \left|\sum_{j=1}^{k} \nu_j\right| + |\nu_{k+1}| \leq \sum_{j=1}^{k} |\nu_j| + |\nu_{k+1}| \leq \sum_{j=1}^{k+1} |\nu_j|$$

(b) Let $\zeta \in \mathbb{C}$. Suppose $0 < |\zeta| < 1$. Then we have

$$\left|\sum_{k=1050}^{4060} \zeta^k\right| \leq \sum_{k=1050}^{4060} |\zeta^k| = \sum_{k=1050}^{4060} |\zeta|^k = |\zeta|^{1050} \cdot \sum_{k=0}^{3010} |\zeta|^k = |\zeta|^{1050} \cdot \frac{1-|\zeta|^{3011}}{1-|\zeta|} < \frac{|\zeta|^{1050}}{1-|\zeta|}.$$

The first inequality is a consequence of Statement (T). The last inequality follows from $|\zeta|^{1050} > 0$ and $0 < |\zeta|^{3011} < 1$.

8. (a) *Hint.* The appropriate proposition $P(n)$ upon which mathematical induction is applied is:

   Suppose $b_1, b_2, \cdots, b_n$ are positive real numbers. Then $(1+b_1)(1+b_2)\cdot \ldots \cdot(1+b_n) > 1+(b_1+b_2+\cdots+b_n)$.

   (b) *Hint.* Be aware that whenever $0 < p < 1$, the inequalities $0 < 1-p < 1$ and $0 < (1-p)(1+p) < 1$ hold.

9. **Solution.**

   (a) Let a, b, u, v be positive real numbers. Suppose $u + v = 1$.
   We have

$$\begin{aligned} a^2u + b^2v - (au+bv)^2 &= a^2u + b^2v - a^2u^2 - b^2v^2 - 2abuv \\ &= a^2u(1-u) + b^2v(1-v) - 2abuv = a^2uv + b^2uv - 2abuv = (a-b)^2uv \geq 0. \end{aligned}$$

   Then $a^2u + b^2v \geq (au+bv)^2$. Therefore $\sqrt{a^2u + b^2v} \geq au + bv$.

   (b) Denote by $P(n)$ the statement below:

   Suppose $c_1, c_2, \cdots, c_n, x_1, x_2, \cdots, x_n$ be positive real numbers. Further suppose $x_1 + x_2 + \cdots + x_n = 1$. Then

$$\sqrt{{c_1}^2x_1 + {c_2}^2x_2 + \cdots + {c_n}^2x_n} \geq c_1x_1 + c_2x_2 + \cdots + c_nx_n.$$

   - By the result in part (a), $P(2)$ is true.
   - Let $k \in \mathbb{N}\backslash\{0,1\}$. Suppose $P(k)$ is true.
     We verify that $P(k+1)$ is true:

     Suppose $c_1, c_2, \cdots, c_k, c_{k+1}, x_1, x_2, \cdots, x_k, x_{k+1}$ be positive real numbers. Further suppose $x_1 + x_2 + \cdots + x_k + x_{k+1} = 1$.
     Define $t = x_1 + x_2 + \cdots + x_k$. We have $t > 0$ and $t + x_{k+1} = 1$.
     For each $j = 1, 2, \cdots, k$, define $u_j = \dfrac{x_j}{t}$. Note that $u_1, u_2, \cdots, u_k$ are positive real numbers and $u_1 + u_2 + \cdots + u_k = 1$.
     Define $d = \sqrt{{c_1}^2u_1 + {c_2}^2u_2 + \cdots + {c_k}^2u_k}$. By definition, $d^2t = {c_1}^2x_1 + {c_2}^2x_2 + \cdots + {c_k}^2x_k$
     By $P(k)$, we have $d \geq c_1u_1 + c_2u_2 + \cdots + c_ku_k$.
     Now

$$\begin{aligned} \sqrt{{c_1}^2x_1 + {c_2}^2x_2 + \cdots + {c_k}^2x_k + {c_{k+1}}^2x_{k+1}} &= \sqrt{d^2t + {c_{k+1}}^2x_{k+1}} \\ &\geq dt + c_{k+1}x_{k+1} \quad \text{(by } P(2)) \\ &= (c_1u_1 + c_2u_2 + \cdots + c_ku_k)t + c_{k+1}x_{k+1} \\ &= c_1x_1 + c_2x_2 + \cdots + c_kx_k + c_{k+1}x_{k+1} \end{aligned}$$

     Hence $P(k+1)$ is true.

   By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}\backslash\{0,1\}$.

10. (a) **Answer.**

   (I) x is an irrational number
   (II) it were true that $\sqrt{x}$ was a rational number
   (III) x is positive
   (IV) $\sqrt{x}$ was a rational number
   (V) a rational number
   (VI) an irrational number
   (VII) assumption
   (VIII) false

   (b) **Solution.**

   i. Let x be a positive real number, r be a positive rational number, and n be an integer greater than 1. Suppose x is an irrational number.
   Further suppose it were true that $\sqrt[n]{x+r}$ was a rational number.
   Write $y = \sqrt[n]{x+r}$. Note that $y^n - r = x$.
   Since y was a rational number, y^n would be a rational number. Moreover, since r is a rational number, $y^n - r$ would be a rational number.
   Therefore x would be a rational number. But by assumption x is an irrational number. Contradiction arises.
   Hence the assumption that $\sqrt[n]{x+r}$ was rational is false. $\sqrt[n]{x+r}$ is an irrational number in the first place.

   ii. Let $r, s, t \in \mathbb{R}$. Suppose r is a non-zero rational number and s is an irrational number.
   Further suppose it were true that both $rs+t, rs-t$ were rational numbers.
   Note that $2rs = (rs+t) + (rs-t)$. Then $2rs$ would be a rational number.
   Since $2, r$ are non-zero rational numbers, $2r$ is a non-zero rational number.
   Note that $s = \dfrac{2rs}{2r}$. Then s would be a rational number.
   But s is an irrational number.
   Contradiction arises.
   Hence at least one of $rs+t, rs-t$ is irrational.

11. (a) **Answer.**

(I) Suppose it were true that $\sqrt[3]{3}$ was not irrational
(II) there would exist some $m, n \in \mathbb{Z}$
(III) $n \neq 0$ and $m = n \cdot \sqrt[3]{3}$
(IV) m^3 would be divisible by 3
(V) Euclid's Lemma
(VI) there would exist some $k \in \mathbb{Z}$ such that $m = 3k$
(VII) Note that $3k^3$ was an integer. Then n^3 would be divisible by 3.
(VIII) 3 is a prime number
(IX) n would be divisible by 3
(X) m, n have no common factors other than $-1, 1$

(b) i. **Solution.**

Suppose it were true that $\sqrt[5]{7}$ was not irrational.
Then $\sqrt[5]{7}$ would be a rational number. There would exist some $m, n \in \mathbb{Z}$ such that $n \neq 0$ and $\sqrt[5]{7} = \dfrac{m}{n}$.
Without loss of generality, we may assume that m, n have no common factors other than $1, -1$.
We would have $m = \sqrt[5]{7}n$. Then $m^5 = 7n^5$.
Now m^5 was divisible by 7. Also note that 7 is a prime number. By Euclid's Lemma, m would be divisible by 7. Therefore there existed some $k \in \mathbb{Z}$ such that $m = 7k$.
Then we would have $7^5k^5 = (7k)^5 = m^5 = 7n^5$. Therefore $n^5 = 7^4k^5 = 7(7^3k^5)$.
Now n^5 was divisible by 7. Also note that 7 is a prime number. By Euclid's Lemma, n would be divisible by 7.
Therefore both m, n would be divisible by 7. 7 would be a common factor of m, n. Recall that we assumed that m, n have no common factors other than $-1, 1$.
Contradiction arises.
Therefore the assumption that $\sqrt[5]{7}$ was not irrational is false. $\sqrt[5]{7}$ is irrational.

ii. ——

12. ——