MATH1050 Exercise 3 Supplement

1. Let z, w be complex numbers. Verify the statements below:

   (a) $\mathsf{Re}(z) = \dfrac{z+\bar{z}}{2}$ and $\mathsf{Im}(z) = \dfrac{z-\bar{z}}{2i}$.

   (b) $\mathsf{Re}(\bar{z}) = \mathsf{Re}(z)$, $\mathsf{Im}(\bar{z}) = -\mathsf{Im}(z)$, and $|\bar{z}| = |z|$.

   (c) $\bar{\bar{z}} = z$.

   (d) $\mathsf{Re}(z+w) = \mathsf{Re}(z) + \mathsf{Re}(w)$ and $\mathsf{Im}(z+w) = \mathsf{Im}(z) + \mathsf{Im}(w)$.

   (e) $\mathsf{Re}(zw) = \mathsf{Re}(z)\mathsf{Re}(w) - \mathsf{Im}(z)\mathsf{Im}(w)$ and $\mathsf{Im}(zw) = \mathsf{Re}(z)\mathsf{Im}(w) + \mathsf{Im}(z)\mathsf{Re}(w)$.

   (f) $\overline{z+w} = \bar{z} + \bar{w}$ and $\overline{zw} = \bar{z}\bar{w}$.

2. Let $\omega = \dfrac{1+i}{\sqrt{2}}$.

   (a) Write down the respective values of ω^2, ω^8, ω^{2016}.

   (b) Hence, or otherwise, find the value of $\left|\sum_{k=0}^{2017} \omega^k\right|^2$. Leave your answer in surd form.

3. Let ζ be a complex number with real part a and modulus r. Express $\zeta^m + \bar{\zeta}^m$ in terms of a, r alone for $m = 1, 2, 3, 4, 5, 6$.

4. Let k be a real number, and ζ be the complex number defined by $\zeta = (2+i)k^2 - 3(1+i)k - 2(1-i)$.

   (a) Express $\mathsf{Re}(\zeta)$ and $\mathsf{Im}(\zeta)$ in terms of k.

   (b) i. Suppose ζ is real. What are the possible values of k and ζ respectively? Justify your answer.

   ii. Suppose ζ is purely imaginary. What are the possible values of k and ζ respectively? Justify your answer.

   iii. Suppose $\mathsf{Re}(\zeta) = \mathsf{Im}(\zeta)$ and $\zeta \neq 0$. What are the possible values of k and ζ respectively? Justify your answer.

5. Let a, b, h, k be real numbers, with $h \neq 0$. Let ω be a complex number, with $|\omega| = 1$. Suppose $a + bi = \dfrac{h}{k+\omega}$.

   (a) Verify that $\omega = \dfrac{(h-ak) - bki}{a+bi}$.

   (b) Hence deduce that $(k^2-1)(a^2+b^2) + h^2 - 2ahk = 0$.

6. Let z, w be complex numbers. Suppose $w \neq 0$.

   (a) Suppose $|z| = |z-w|$. Prove that $\mathsf{Re}\left(\dfrac{z}{w}\right) = \dfrac{1}{2}$.

   (b) Suppose $|z| = |z-w| = |w|$. Express z in terms of w.

7. Let α, β be complex numbers. Suppose $|\alpha| = |\beta| = |\alpha+\beta| = 1$.

   (a) Find the value of $(\alpha+\beta)\left(\dfrac{1}{\alpha} + \dfrac{1}{\beta}\right)$.

   (b) Hence, or otherwise, prove that $\alpha^3 = \beta^3$.

   Remark. Start by considering the expression $\alpha^3 - \beta^3$: can you factorize it?

8. Let z, w be complex numbers.

   (a) Verify that $zw + \bar{z}\bar{w}$ is a real number.

   (b) Hence deduce that $zw + \bar{z}\bar{w} \leq 2|z| \cdot |w|$.

9. Let z, w be complex numbers. Prove that $|1 - z\bar{w}|^2 - |z-w|^2 = (1-|z|^2)(1-|w|^2)$.

10. Let a, b be real numbers. Suppose $a + bi = \dfrac{2+4i}{1-i}(b+i)$. Determine the values of a, b respectively.

11. Let a, b be real numbers. Consider the quadratic equation $z^2 + az + b = 0$ with unknown z. Suppose $z = 1+i$ is a solution of this equation. Find the values of a, b.

12. Let a, b, c, d be real numbers, and $f(x)$ be the quadratic polynomial with complex coefficients given by $f(x) = x^2 + (a+bi)x + (c+di)$.

Prove the statements below:

(a)$^\diamondsuit$ *$f(x)$ has a pair of distinct real root iff ($b = d = 0$ and $a^2 - 4c > 0$).*

(b)$^\diamondsuit$ *$f(x)$ has a pair of distinct complex roots which are conjugate to each other iff ($b = d = 0$ and $a^2 - 4c < 0$).*

(c)$^\clubsuit$ *$f(x)$ has one real root and one non-real root iff ($b \neq 0$ and $d^2 - abd + b^2c = 0$).*

13. (a) Let ζ be a complex number. Suppose $\zeta^2 = \bar{\zeta}$.

i. Prove that $|\zeta| = 0$ or $|\zeta| = 1$.

ii. Hence, or otherwise, prove that $\zeta = 0$ or $\zeta = 1$ or $\zeta^2 + \zeta + 1 = 0$.

(b) Solve the equation $z^2 = \bar{z}$ with unknown z in $\mathbb{C}$.

14.$^\diamondsuit$ Solve the system of equations

$$\begin{cases} |1+z| &= |3-z| \\ z\bar{z} &= 4 \end{cases}$$

with unknown z in $\mathbb{C}$.

Remark. If you are not required to display any algebraic manipulation, can you write down the answer quickly by simply considering the geometry on the Argand plane?

15.$^\diamondsuit$ Consider the curve C on the Argand plane defined by the equation $z\bar{z} - (2+3i)z - (2-3i)\bar{z} = 12$.

Let $p = 4 - 5i$. Determine the points on the curve C on the Argand plane which are respectively of the maximum distance and of the minimum distance from the point p.

Remark. Try to re-write the equation which defines C in such a way that you can identify C as some kind of curves familiar to you.

16.$^\diamondsuit$ Let $w = 4 + 3i$, and C be the curve on the Argand plane defined by the equation $|z - 3i| = 2$. Determine the point(s) in C at minimum distance from w on the Argand plane. Justify your answer.

Remark. If you are not required to display any algebraic manipulation, can you write down the answer quickly by simply considering the geometry on the Argand plane?

17.$^\diamondsuit$ Consider the infinite straight line ℓ defind by the equation $|z - (3+i)| = |z - (5+5i)|$ with unknown z in the complex numbers.

(a) Re-express the equation in the form $\mathsf{Im}(z) = a\mathsf{Re}(z) + b$, in which a, b are real numbers. You have to give explicit values for a, b respectively.

(b) What is the smallest possible value of $|z|$ if z lies on ℓ?

Remark. If you are not required to display any algebraic manipulation, can you write down the answer quickly by simply considering the geometry on the Argand plane?

18. Express the complex numbers $-1-i$, $1-i$ in polar form. Hence simplify $\dfrac{-1-i}{(1-i)^5}$.

19. Let c be a real number, $z = 1+i$, $w = (c+4) + (c-4)i$. Denote the angle between the line segment joining 0 and z and that joining 0 and w is θ. Suppose $\cos(\theta) = -\dfrac{3}{5}$. Find the possible value(s) of c.

20. Let p, q, r be three distinct non-zero complex numbers. Denote the points in the Argand plane represented by p, q, r by P, Q, R respectively. Denote the origin in the Argand plane by O. Suppose $OPQR$ is a rectangle in the Argand plane. Suppose $|p| = 2$, and $\arg(p) = \dfrac{\pi}{6}$. Suppose $|q - p| = 1$. Also suppose $\mathsf{Re}(r) < 0$.

(Draw an appropriate diagram before you proceed any further.)

(a) Express p, r in standard form.

(b) Hence find q. Express your answer in standard form.

21. (a) Let α, σ be complex numbers.

i.$^\diamondsuit$ Suppose $\mathsf{Re}(\alpha) = a$, $\mathsf{Im}(\alpha) = b$, $\mathsf{Re}(\sigma) = s$, $\mathsf{Im}(\sigma) = t$.
Prove that $\alpha = \sigma^2$ iff ($a = s^2 - t^2$ and $b = 2st$).

ii. Now suppose $|\alpha| = 1$ and $\alpha = \sigma^2$.

A. Prove that $|\sigma| = 1$.

B. Prove that $|\alpha - \sigma| = |\sigma - 1|$.

Remark. Can you give a geometric interpretation of these results on the Argand plane?

(b) Find the square roots of each of the complex numbers below:

i. i

ii. $-i$

iii. $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$

iv. $\frac{1}{2} + \frac{\sqrt{3}}{2}i$

v. $1 + 3i$

vi. $1 + i$

22. Solve for all complex solutions of each of the equations below:

(a) $z^2 + 4iz = 0$.

(b) $z^2 + (3 - 2i)z - 6i = 0$.

(c) $z^2 - 2iz + 1 = 0$.

(d) $z^2 - 4z + (4 - i) = 0$.

(e) $z^2 - (2 + 2i)z + (-4 + 2i) = 0$.

(f) $z - 5i - \frac{6}{z} = 0$.

23. (a) Find the square roots of each of the complex numbers below:

i. $4 + 3i$

ii. $4 - 3i$

(b) Hence, or otherwise, solve for all complex solutions of the equation $x^4 - 8x^2 + 25 = 0$.

24. Let α be a non-zero complex number. Suppose $\alpha^5/\overline{\alpha}^3 = 4$. Also suppose that α is neither real nor purely imaginary.

(a) What is the modulus of α?

(b) Find all possible values of α. Express your answers in standard form.

25. (a)$^\diamondsuit$ Prove the statement ($\sharp$) below:

($\sharp$) *Let z, w be complex numbers. Suppose $z - w \neq 0$. $|z| = |w|$ iff* $\mathsf{Re}\left(\frac{z + w}{z - w}\right) = 0$.

Remark. Can you give a geometric interpretation for the statement ($\sharp$)?

(b) Hence, or otherwise, prove the statement ($\flat$) below:

($\flat$) *Let ζ, μ, ν be complex numbers. Suppose $\mu \neq \nu$. Then $|\zeta - \mu| = |\zeta - \nu|$ iff* $\mathsf{Re}\left(\frac{\zeta - (\mu + \nu)/2}{\mu - \nu}\right) = 0$.

Remark. Can you give a geometric interpretation for the statement ($\flat$) when you regard ζ as a 'variable point' on the Argand plane?

26. Let $z_1, z_2, z_3 \in \mathbb{C}$. Suppose z_1, z_2, z_3 are the three vertices of an isosceles triangle on the Argand plane, in such a way that the equality $|z_1 - z_2| = |z_1 - z_3|$ holds and the angle subtended by the segment joining z_1, z_2 and that joining z_1, z_3 is α.

(a)$^\diamondsuit$ Prove that $(z_2 - z_1)^2 + (z_3 - z_1)^2 = A(z_2 - z_1)(z_3 - z_1)\cos(\alpha)$. Here A is an integer whose value you have to determine explicitly.

Remark. Start by relating the α with $z_2 - z_1$, $z_3 - z_1$.

(b) i. Suppose the isosceles triangle is actually a right-angle triangle. What is the value of $(z_2 - z_1)^2 + (z_3 - z_1)^2$? Justify your answer.

ii. Suppose the isosceles triangle is actually an equilateral triangle. Prove that $z_1^2 + z_2^2 + z_3^2 = Bz_2z_3 + Cz_3z_1 + Dz_1z_2$. Here B, C, D are integers whose respective values you have to determine explicitly.

27.$^\clubsuit$ Let $z_1, z_2, z_3 \in \mathbb{C}$. Suppose z_1, z_2, z_3 are pairwise distinct and $|z_1| = |z_2| = |z_3|$. Prove that the statements below are logically equivalent:

($\dagger$) $z_1 + z_2 + z_3 = 0$.

($\ddagger$) $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$.

Remark. Hence z_1, z_2, z_3 are the three vertices of an equilateral triangle iff $z_1 + z_2 + z_3 = 0$.

28. Let $z \in \mathbb{C}$.

    (a)$^\diamondsuit$ Prove that the statements $(\dagger), (\ddagger)$ are logically equivalent:

    ($\dagger$) *There exists some $\theta \in \mathbb{R}$ such that $z = 1 + \cos(\theta) + i\sin(\theta)$.*

    ($\ddagger$) $|z - 1| = 1$.

    (b)$^\clubsuit$ Prove that the following statements are logically equivalent:

    ($\dagger$) *There exists some $\theta \in \mathbb{R}$ such that $\cos(\theta) \neq -1$ and $z = \dfrac{1}{1 + \cos(\theta) + i\sin(\theta)}$.*

    ($\ddagger$) $\mathsf{Re}(z) = \dfrac{1}{2}$.

29. Let $u, v \in \mathbb{C}$.

    (a)$^\diamondsuit$ Suppose $u\bar{v} \in \mathbb{R}$. Prove the following statements:

    i. There exist some $\alpha, \beta \in \mathbb{R}$, not both zero, such that $\alpha u + \beta v = 0$.

    ii. $|u| + |v| = \begin{cases} |u + v| & \text{if} \quad u\bar{v} \geq 0 \\ |u - v| & \text{if} \quad u\bar{v} < 0 \end{cases}$.

    (b)$^\clubsuit$ Suppose $u\bar{v} \notin \mathbb{R}$. Prove that for any $z \in \mathbb{C}$, there exist some unique $\alpha, \beta \in \mathbb{R}$ such that $z = \alpha u + \beta v$.

    Remark. What you are required to prove is that under the assumption '$u\bar{v} \notin \mathbb{R}$', this pair of 'logically independent' statements holds simultaneously:

    ($\dagger$) *For any $z \in \mathbb{C}$, there exist some $\alpha, \beta \in \mathbb{R}$ such that $z = \alpha u + \beta v$.*

    ($\dagger$) *For any $z \in \mathbb{C}$, for any $\alpha, \beta, \alpha', \beta' \in \mathbb{R}$, if ($z = \alpha u + \beta v$ and $z = \alpha' u + \beta' v$) then ($\alpha = \alpha'$ and $\beta = \beta'$).*