MATH1050 Exercise 3

1. (a) Suppose ζ is a complex number. Verify that $|\zeta|^2 = \zeta\bar{\zeta}$.

   (b) Let z, w be complex numbers. Suppose $w \neq 0$. Verify that $\dfrac{z}{w} = \dfrac{z\bar{w}}{|w|^2}$.

   Remark. This looks easy. And it is indeed easy. But it suggests division involving complex numbers can be done in a much more comfortable way. (How?)

2. Let z, w be complex numbers.

   (a) Verify that $|zw| = |z||w|$.

   (b) Suppose $z \neq 0$ and $w \neq 0$, and θ, φ are respective arguments of z, w. Verify that $zw = |zw|(\cos(\theta+\varphi) + i\sin(\theta+\varphi))$.

3. Let $\omega = \dfrac{\sqrt{3}+i}{2}$.

   (a) Write down the respective values of ω^2, ω^3, ω^{11}, ω^{12}.

   (b) Hence, or otherwise, find the value of $\displaystyle\sum_{k=0}^{2230} \omega^{k+1}$.

4. Let a, b, c be real numbers. Suppose $a^2 + b^2 + c^2 = 1$ and $c \neq 1$. Define $z = \dfrac{a+bi}{1-c}$.

   (a) Express $|z|^2$ in terms of c alone.

   (b) Express each of a, b, c in terms of $z, \bar{z}$ alone.

5. (a) Prove the statement below, known as the **Parallelogram Identity**:

   - *Suppose z, w are complex numbers. Then $|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2$.*

   (b) Prove the statements below:

   i. *Suppose r, s, t are complex numbers. Then $|2r-s-t|^2 + |2s-t-r|^2 + |2t-r-s|^2 = 3(|s-t|^2 + |t-r|^2 + |r-s|^2)$.*

   ii. *Let ζ, α, β be complex numbers. Suppose $\zeta^2 = \alpha^2 + \beta^2$. Then $|\zeta+\alpha| + |\zeta-\alpha| = |\zeta+\beta| + |\zeta-\beta|$.*

6. Let $z \in \mathbb{C}$. Suppose $4|z+1| = |z+16|$. Verify that z lies on the circle with centre at 0 and with radius 4 (on the Argand plane).

7.$^\diamond$ Write $\mu = 1+i$, $\nu = 3-i$.

   (a) Let $\eta \in \mathbb{C}$. Prove that the point η lies on the ('infinite') straight line joining μ, ν iff $\mathsf{Re}(\eta) + \mathsf{Im}(\eta) - 2 = 0$.

   (b) Consider the curve C on the Argand plane defined by the equation $|z-\mu| = |z-\nu|$.

   i. Prove that the curve C is also described by the equation $\mathsf{Re}(z) - \mathsf{Im}(z) - 2 = 0$.

   ii. Hence, or otherwise, prove that the curve C is the perpendicular bisector for the line segment joining μ, ν (on the Argand plane).

   Remark. Given the same straight line, there are various ways to write down an equation (with unknown z in the complex numbers) whose solution set is the straight line concerned on the Argand plane:

   - $a\mathsf{Re}(z) + b\mathsf{Im}(z) + c = 0$, in which a, b, c are given real number with a, b being not both zero.
   - $\alpha\bar{z} + \bar{\alpha}z + d = 0$, in which α is a given non-zero complex number and d is a given real number.
   - $|z-\beta| = |z-\gamma|$, in which β, γ are given distinct complex numbers.

8. *There is no need to give any justifications for your answers in this question.*

   Find all the solutions of the system of equations $\begin{cases} |z-2-2i| &= 2 \\ |z-4+2i| &= |z-2i| \end{cases}$ with unknown z in $\mathbb{C}$.

   Remark. What are the curves described by the respective equations in the Argand plane?

9. *There is no need to give any justifications for your answers in this question.*

   Consider the system of equations $(S_{\alpha,r}) : \begin{cases} |z-2i| &= 2 \\ |z-4-4i| &= |z| \\ |z-\alpha| &= r \end{cases}$ with unknown z in $\mathbb{C}$. Here α is some complex number and r is a non-negative real number.

   Suppose that $(S_{\alpha,r})$ has two distinct solutions.

   (a) Write down all solutions of $(S_{\alpha,r})$.

   (b) What is the smallest possible value of r?

   (c) What is the value of α if $|\mathsf{Re}(\alpha)| = |\mathsf{Im}(\alpha)|$?

Remark. What are the curves described by the respective equations in the Argand plane?

10. Let ω be a complex number. Suppose $|\omega| = 1$ and $\mathsf{Im}(\omega) \geq 0$. Further suppose $\omega^2 + \frac{5}{\omega} - 2$ is purely imaginary. Find the value of ω.
Remark. Use the polar form of ω.

11. Let α, β, γ be real numbers. Suppose $\alpha + \beta + \gamma = 2\pi$.
Define λ, μ, ν by $\lambda = \cos(\alpha) + i\sin(\alpha)$, $\mu = \cos(\beta) + i\sin(\beta)$, $\nu = \cos(\gamma) + i\sin(\gamma)$ respectively.
    (a) Find the value of $\lambda\mu\nu$.
    (b) Prove that $\cos(\alpha) = \frac{1}{2}(\lambda + \frac{1}{\lambda})$ and $\cos(2\alpha) = \frac{1}{2}(\lambda^2 + \frac{1}{\lambda^2})$.
    (c)$^\diamondsuit$ Hence, or otherwise, prove that $\cos(2\alpha) + \cos(2\beta) + \cos(2\gamma) = 4\cos(\alpha)\cos(\beta)\cos(\gamma) - 1$.

12. (a)$^\diamondsuit$ Prove the statements below:
    i. *Suppose z is a complex number. Then $|\mathsf{Re}(z)| \leq |z|$. Moreover, equality holds iff z is real.*
    ii. *Suppose z be a complex number. Then $|\mathsf{Im}(z)| \leq |z|$. Moreover, equality holds iff z is purely imaginary.*

    (b)$^\diamondsuit$ Prove the statement below:
    - *Suppose u, v are complex numbers. Then $|\mathsf{Re}(u\bar{v})| \leq |u||v|$. Moreover, equality holds iff there exist some real numbers p, q such that $pu + qv = 0$ and p, q are not both zero.*

    (c)$^\clubsuit$ Prove the statements below (known collectively as the **Triangle Inequality for complex numbers**):
    i. *Suppose z, w are complex numbers. Then $|z + w| \leq |z| + |w|$. Moreover, equality holds iff there exist some non-negative real numbers s, t such that $sz = tw$ and s, t are not both zero.*
    ii. *Suppose u, v are complex numbers. Then $|\,|u| - |v|\,| \leq |u - v|$. Moreover, equality holds iff there exist some non-negative real numbers h, k such that $hu = kv$ and h, k are not both zero.*

13. We introduce the definitions below:
    - *Let $z \in \mathbb{C}$. The number z is said to be a* **Gaussian integer** *if both of $\mathsf{Re}(z)$, $\mathsf{Im}(z)$ are integers.*
    - *The set of all Gaussian integers is denoted by $\mathbb{G}$.*

    (a) Prove the statements below:
    i. *Suppose $s \in \mathbb{Z}$. Then $s \in \mathbb{G}$.*
    ii. *Let s is a Gaussian integer. Suppose $s \neq 0$. Then $|s| \geq 1$.*
    iii. *Suppose $s, t \in \mathbb{G}$. Then $\bar{s} \in \mathbb{G}$, $s + t \in \mathbb{G}$ and $st \in \mathbb{G}$.*
    iv. *Suppose $s, t \in \mathbb{G}$. Further suppose $|st| = 1$. Then $|s| = |t| = 1$.*

    (b) We introduce the definition below:
    - *Let $u, v \in \mathbb{G}$. The number u is said to be* **$\mathbb{G}$-divisible** *by v if there exists some $s \in \mathbb{G}$ such that $u = sv$.*

    Prove the statements below:
    i. *Suppose $u \in \mathbb{G}$. Then u is $\mathbb{G}$-divisible by u.*
    ii.$^\diamondsuit$ *Let $u, v \in \mathbb{G}$. Suppose (u is $\mathbb{G}$-divisible by v and v is $\mathbb{G}$-divisible by u). Then $|u| = |v|$.*
    iii. *Let $u, v, w \in \mathbb{G}$. Suppose (u is $\mathbb{G}$-divisible by v and v is $\mathbb{G}$-divisible by w). Then u is $\mathbb{G}$-divisible by w.*
    iv. *Let $u, v, t \in \mathbb{G}$. Suppose (u is $\mathbb{G}$-divisible by t and v is $\mathbb{G}$-divisible by t). Then $u + v$ is $\mathbb{G}$-divisible by t.*
    v. *Let $u, v, t \in \mathbb{G}$. Suppose (u is $\mathbb{G}$-divisible by t or v is $\mathbb{G}$-divisible by t). Then uv is $\mathbb{G}$-divisible by t.*
    vi. *Let $u, v \in \mathbb{G}$. Suppose $u \neq 0$ and u is $\mathbb{G}$-divisible by v. Then $|v| \leq |u|$.*

    (c)$^\diamondsuit$ We introduce the definition below:
    - *Let $z \in \mathbb{C}$. The number z is said to be a* **Gaussian rational** *if there exist some Gaussian integer u, v such that $v \neq 0$ and $u = vz$.*

    Prove the statements below:
    i. *Suppose z is a rational number. Then z is a Gaussian rational.*
    ii. *Suppose z is a Gaussian integer. Then z is a Gaussian rational.*
    iii. *Suppose $z \in \mathbb{C}$. Then z is a Gaussian rational iff (there exist some $s, t \in \mathbb{Q}$ such that $z = s + ti$.)*
    iv. *Suppose z, w are Gaussian rationals. Then $z + w$ is a Gaussian rational.*
    v. *Suppose z, w are Gaussian rationals. Then $z - w$ is a Gaussian rational.*
    vi. *Suppose z, w are Gaussian rationals. Then zw is a Gaussian rational.*
    vii. *Suppose z, w are Gaussian rationals. Further suppose $w \neq 0$. Then $\frac{z}{w}$ is a Gaussian rational.*

Remark. *Preview* on your *algebra* course. The set $\mathbb{G}$ is a subset of $\mathbb{C}$ and contains $\mathbb{Z}$ as a subset. In your *algebra* course, the set $\mathbb{G}$ is likely denoted by $\mathbb{Z}[i]$. The set $\mathbb{G}$, together with addition and multiplication for complex numbers, constitutes an *integral domain*; further together with the modulus for complex numbers, it constitutes an *Euclidean domain*. It behaves in many ways similar to integers, and similar to polynomials with real coefficients. Many results on the notion of '*divisibility*' for integers and polynomials with real coefficients in school maths have their analogues for Gaussian integers. The set of all Gaussian rationals is a subset of $\mathbb{C}$ and contains each of $\mathbb{Q}$ and $\mathbb{G}$ as a subset. In your *algebra* course, it is likely denoted by $\mathbb{Q}[i]$ in some contexts, and by $\mathbb{Q}(i)$ in some other contexts. This set, together with addition and multiplication for complex numbers, constitutes a *field*.