MATH1050 Exercise 2 Supplement

1. Let a, b, c, p, q, r be numbers, with $a \neq 0$ and $p \neq 0$. Let $f(x), g(x)$ be the quadratic polynomials given by $f(x) = ax^2 + bx + c$, $g(x) = px^2 + qx + r$ respectively. Suppose α, β are the roots of $f(x)$, and $\alpha \neq 0$, $\beta \neq 0$. Further suppose $\frac{1}{\alpha}, \frac{1}{\beta}$ are the roots of $g(x)$.

   Prove that $ap = cr$ and $aq = br$.

2. Let p, q, r be numbers. Suppose $p + q - 2r \neq 0$. Let $f(x)$ be the quadratic polynomial given by $f(x) = (p + q - 2r)x^2 + (q + r - 2p)x + (r + p - 2q)$.

   (a) Verify that 1 is a root of $f(x)$.

   (b) Prove that $f(x)$ has a repeated root iff $q = r$.

3. Let k be a number, and $f(x)$ be the quadratic polynomial given by $f(x) = x^2 - 2x + k$. Suppose α, β are the roots of $f(x)$.

   (a) Find the quadratic polynomial $g(x)$ with leading coefficient 1 whose roots are α^3, β^3.

   (b) Also prove that the discriminant Δ_g of $g(x)$ is given by $\Delta_g = A(B - k)(C - k)^2$. Here A, B, C are integers, whose values you have to determine explicitly.

   (c) Suppose k is a real number. Further suppose α, β are not real numbers, but α^3, β^3 are real numbers. Find all possible values of k. Justify your answer.

4. Let a, b be real numbers. Suppose $a > b$ and $a + b \neq 0$. Let $f(x)$ be the quadratic polynomial given by $f(x) = (a - b)x^2 - 2(a^2 + b^2)x + (a^3 - b^3)$.

   (a) Prove that the roots of $f(x)$ are real and distinct iff $ab > 0$.

   (b) Suppose α, β are distinct real roots of $f(x)$, and $\alpha > \beta$. Prove that $\alpha - \beta = \dfrac{2(a + b)\sqrt{ab}}{a - b}$.

5.$^\diamondsuit$ Let a, b be real numbers. Suppose $a > b > 0$. Let $f(x)$ be the quadratic polynomial given by $f(x) = 2x^2 - (3a + b)x + ab$. Prove that $f(x)$ has two distinct real roots, one of them greater than b and the other less than b.

6. Let $a, b, x \in \mathbb{R}$. Suppose $a < b$. Prove the statements below:

   (a) *Suppose* $(x - a)(x - b) > 0$. *Then* $x < a$ *or* $x > b$.

   (b) *Suppose* $(x - a)(x - b) < 0$. *Then* $a < x < b$.

   Remark. In fact, the converses of the respective statements are also true. These statements and their respective converses constitute the theoretical foundation in the arithmetic for solving quadratic inequalities. Also deduce on your own the analogous theoretical results for non-strict inequalities.

7. (a)$^\diamondsuit$ Let $u, v, w \in \mathbb{R}$. Suppose $u < v < w$. Prove the statements below

   i. *Suppose* $uvw > 0$. *Then* $0 < u < v < w$ or $u < v < 0 < w$.

   ii. *Suppose* $uvw < 0$. *Then* $u < 0 < v < w$ or $u < v < w < 0$.

   Remark. At some stage of the argument we may come into situations in where we use the *Distributive Laws for 'and', 'or'* (with or without our being aware of them).[1] Also note that the converses of the respective statements are also true.

   (b)$^\clubsuit$ Let $a, b, c, x \in \mathbb{R}$. Suppose $a < b < c$. Prove the statements below:

   i. *Suppose* $(x - a)(x - b)(x - c) > 0$. *Then* $a < x < b$ *or* $x > c$.

   ii. *Suppose* $(x - a)(x - b)(x - c) < 0$. *Then* $x < a$ *or* $b < x < c$.

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[1] The Distributive Laws for 'and', 'or' in logic may be in-formally stated as below:

A. *The pair of statements below are the same in the sense that one holds exactly when the other holds:*
- *(blah-blah-blah* or *bleh-bleh-bleh)* and *bloh-bloh-bloh.*
- *(blah-blah-blah* and *bloh-bloh-bloh)* or *(bleh-bleh-bleh* and *bloh-bloh-bloh).*

B. *The pair of statements below are the same in the sense that one holds exactly when the other holds:*
- *(blah-blah-blah* and *bleh-bleh-bleh)* or *bloh-bloh-bloh.*
- *(blah-blah-blah* or *bloh-bloh-bloh)* and *(bleh-bleh-bleh* or *bloh-bloh-bloh).*

More will be said of them in the discussion on *logic*.

Remark. The respective converses of the statements are also true. The statements and their respective converses suggest how we may proceed with the arithmetic for solving cubic inequalities (as long as we know how to factorize the cubic polynomials involved). Deduce on your own the analogous theoretical results for non-strict inequalities.

8. Solve for all real solutions of each of the inequalities (or systems of inequalities) below.

(a) $x^2 \geq 5x - 6$.
(b) $(x-2)(x+3) < 2(x-2)$.
(c) $(x+8)(2x-3) < (x-5)(x+8)$.
(d) $(x-1)(x-2)(x-3) \geq 27x - 6$.
(e) $(x-1)^2(x-4) \geq 0$.
(f) $(x-1)(x-3)^2 \leq 0$.
(g) $(x+3)x(x-1)(x-2) > 0$.
(h) $(x-1)(x-2)(x-4)(x-8) \leq 0$.

Remark. Now suppose you are not required to give any step of algebraic manipulation. Can you modify the 'graphical method' for solving equations in *school mathematics* to determine the answer for each part as quickly as possible?

9.$^\diamond$ Solve for all real solutions of each of the inequalities (or systems of inequalities) below.

(a) $x > -\dfrac{5}{x} + 6$.
(b) $x \leq -\dfrac{6}{x+1} + 4$.
(c) $2x - 1 \leq \dfrac{3}{x-1} - 4$.
(d) $\dfrac{2x}{x+1} \geq 2x - 1$.
(e) $\dfrac{2x-3}{x+1} \leq 1$.
(f) $\dfrac{3x+1}{x+2} \geq 1$.
(g) $\dfrac{1}{x+1} \leq \dfrac{1}{3-x}$.
(h) $\dfrac{1}{x^2-6x+8} \geq 0$.
(i) $\dfrac{3}{x^2-6x+8} \geq 1$.
(j) $\dfrac{x^2-7x+12}{x^2-3x+2} \leq 0$.
(k) $\dfrac{x^2-7x+12}{x^2-3x+2} \leq -1$.
(l) $\dfrac{x^2-1}{x^2-4} \geq 0$.
(m) $\dfrac{x^2-1}{x^2-4} \geq 1$.

Remark. Now suppose you are not required to give any step of algebraic manipulation. Can you modify the 'graphical method' for solving equations in *school mathematics* to determine the answer for each part as quickly as possible?

10. Solve for all real solutions of each of the inequalities (or systems of inequalities) below:

(a) $|x+3| < 2$.
(b) $|2x-9| \leq 15$.
(c) $|8-3x| \leq 7$.
(d) $|x-2| > 4$.
(e) $|2x+5| \geq 13$.
(f) $|6-x| \geq 6$.
(g) $|x^2+7x-1| < 7$.
(h) $|2x^2-8x-1| \leq 9$.
(i) $|-x^2+2x+3| \geq 5$.
(j) $|x^2-x-3| < 3$.
(k) $\left|\dfrac{3x-1}{4x+1}\right| > 0$.
(l)$^\diamond$ $|\ 2|x| - 9\ | \leq 5$.
(m) $x^2 < |x+2|$.
(n) $|3x+1| \geq x^2+1$.
(o)$^\diamond$ $\dfrac{|x-3|}{2x} < 1$.
(p)$^\diamond$ $\dfrac{|x-9|}{3x+1} > 1$.
(q) $|4x+1| > |x-3|$.
(r)$^\diamond$ $(x+2)|x-2| < -5$.
(s)$^\diamond$ $x^2 - |x| - x < 0$.

11.$^\diamond$ Solve for all real solutions of the inequalities below:

(a) $\sqrt{4x+1} < x+1$.
(b) $\sqrt{6x+3} > 3x+1$.

Remark. The absolute value is implicitly involved in these inequalities: what do you obtain when you square both sides of each inequalities?

12. Let c be a real number. Let $f(x)$ be the polynomial given by $f(x) = (c-4)x^2 + (2c-1)x + (4c-1)$.
Suppose α, β are the roots of $f(x)$, and $\alpha < 0 < \beta$.

(a) By considering the product $\alpha\beta$, or otherwise, prove that $\dfrac{1}{4} < c < 4$.
(b) Further suppose $\alpha + \beta < 0$. Prove that $\dfrac{1}{4} < c < \dfrac{1}{2}$.

13. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{x^2-2x+3}{x^2+2x+3}$ for any $x \in \mathbb{R}$.

(a) Let $\alpha \in \mathbb{R}$. Prove that $2 - \sqrt{3} \leq f(\alpha) \leq 2 + \sqrt{3}$.

Remark. There is no need to use calculus. Write $\beta = f(\alpha)$ and re-express the equality $f(\alpha) = \dfrac{\alpha^2-2\alpha+3}{\alpha^2+2\alpha+3}$ in the form $A\alpha^2 + B\alpha + C = 0$. Then ask what you have learnt about quadratic equations will tell you.

(b) Prove that f attains absolute minimum value $2-\sqrt{3}$ and attains absolute maximum value $2+\sqrt{3}$.

14. Let $f : \mathbb{R}\backslash\{-1\} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{x^2+x+1}{x+1}$ for any $x \in \mathbb{R}\backslash\{-1\}$.

   (a) Let $\alpha \in \mathbb{R}\backslash\{-1\}$. Prove that $f(\alpha) \leq -3$ or $f(\alpha) \geq 1$.

   Remark. There is no need to use calculus. Write $\beta = f(\alpha)$ and re-express the equality $\beta = \dfrac{\alpha^2+\alpha+1}{\alpha+1}$ in the form $A\alpha^2 + B\alpha + C = 0$. Then ask what you have learnt about quadratic equations will tell you.

   (b)$^\diamond$ Does f attain the values -3, 1? Justify your answer.

15. Let $f : \mathbb{R}\backslash\{2,4\} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{(x-1)(x-5)}{(x-2)(x-4)}$ for any $x \in \mathbb{R}\backslash\{2,4\}$.

   (a) Let $\alpha \in \mathbb{R}\backslash\{2,4\}$. Prove that $f(\alpha) \leq 1$ or $f(\alpha) \geq 4$.

   Remark. There is no need to use calculus. Write $\beta = f(\alpha)$ and re-express the equality $\beta = \dfrac{(\alpha-1)(\alpha-5)}{(\alpha-2)(\alpha-4)}$ in the form $A\alpha^2 + B\alpha + C = 0$. Then ask what you have learnt about quadratic equations will tell you.

   (b)$^\diamond$ Does f attain the value 1? Justify your answer.

   (c)$^\diamond$ Does f attain the value 4? Justify your answer.

16. Let n be a positive integer.

   (a) Prove that $k(n-k+1) \geq n$ for each integer k amongst $1, 2, \cdots, n$.

   (b) Hence, or otherwise, prove that $(n!)^2 \geq n^n$.

17. (a) Let $n \in \mathbb{N}\backslash\{0\}$. Prove that $\dfrac{2n}{2n+1} < \dfrac{2n+1}{2n+2}$.

   Remark. There is no need for mathematical induction.

   (b) Prove that $\displaystyle\prod_{k=1}^{5000} \frac{2k-1}{2k} < \frac{1}{100}$

18. (a) Prove the statement ($\sharp$) below:

   ($\sharp$) *Suppose $x, y \in \mathbb{R}$. Then $x^2 + y^2 \geq 2xy$. Moreover, equality holds iff $x = y$.*

   (b) Prove the statement ($\flat$) below:

   ($\flat$) *Suppose $u, v, w \in \mathbb{R}$. Then $u^2 + v^2 + w^2 \geq uv + vw + wu$. Moreover equality holds iff $u = v = w$.*

   (c) By applying the result described by ($\sharp$), or otherwise, prove the statements below:

   i. *Suppose r, s, t be positive real numbers. Then $r + s + t \geq \sqrt{rs} + \sqrt{st} + \sqrt{tr}$.*

   ii. *Suppose $x, y, z \in \mathbb{R}$. Then $x^2y^2 + y^2z^2 + z^2x^2 \geq xyz(x + y + z)$.*

   iii. *Suppose a, b, c, d are positive real numbers. Then $(a+b)(a+c)(a+d)(b+c)(b+d)(c+d) \geq 64(abcd)^{3/2}$.*

   iv. *Let p, q, r, s, t be positive real numbers. Suppose $pqrst = 1$. Then $(1+p)(1+q)(1+r)(1+s)(1+t) \geq 32$.*

19. (a) Prove the statement below:

   ($\sharp$) *Suppose x, y are positive real numbers. Then $\dfrac{x}{y} + \dfrac{y}{x} \geq 2$. Moreover, equality holds iff $x = y$.*

   (b) By applying the result described by ($\sharp$), or otherwise, prove the statements below:

   i. *Suppose $a > 1$ and $b > 1$. Then $\log_a(b) + \log_b(a) \geq 2$. Equality holds iff $a = b$.*

   ii. *Suppose $u \in \mathbb{R}$. Then $\dfrac{u^2+2}{\sqrt{u^2+1}} \geq 2$. Equality holds iff $u = 0$.*

   iii. *Suppose $v \in \mathbb{R}$. Then $\dfrac{v^2}{1+v^4} \leq \dfrac{1}{2}$. Equality holds iff ($v = 1$ or $v = -1$).*

20. We introduce this definition below:

   Let a, b, c be three positive real numbers (not necessarily distinct from each other). The numbers a, b, c are said to **constitute the three sides of a triangle** *if the three inequalities $a+b > c$, $b+c > a$, $c+a > b$ hold simultaneously.*

   (a) Let a, b be positive real numbers. Suppose $a \geq b$. Prove that there exists some positive real number c such that a, b, c constitute the three sides of a triangle.

   Remark. For the geometric interpretation, see Proposition 22, Book I of *Euclid's Elements*.

(b) Let a, b, c be positive real numbers. Suppose a, b, c constitute the three sides of a triangle. Prove that $\sqrt{a}, \sqrt{b}, \sqrt{c}$ constitute the three sides of a triangle.

(c) Let a, b, c be positive real numbers. Suppose a, b, c constitute the three sides of a triangle. Prove the statements below:

i. $a^2 + b^2 + c^2 < 2(ab + bc + ca)$.

ii. $3(ab + bc + ca) \leq (a + b + c)^2 < 4(ab + bc + ca)$.

iii. $(a + b + c)(a + b - c) < 4ab$.

21. *In this question, you may assume without proof the validity of the statement below:*

- *For any real numbers μ, ν, if $0 < \mu < \nu < \frac{\pi}{2}$ then $0 < \sin(\mu) < \sin(\nu) < 1$.*

Let the angles at vertices A, B, C in $\triangle ABC$ be α, β, γ respectively. Suppose each angle in $\triangle ABC$ is an acute angle. Prove the statements below:

(a) $\cos(\frac{\gamma}{2}) > \sin(\frac{\gamma}{2})$.

(b) $\sin(\alpha) + \sin(\beta) > \cos(\alpha) + \cos(\beta)$.

(c) $\sin(\alpha) + \sin(\beta) + \sin(\gamma) > \cos(\alpha) + \cos(\beta) + \cos(\gamma)$.

22. We introduce the definitions below:

- *Let $a, b \in \mathbb{R}$.*

We define the **maximum** *of a, b, which we denote by $\max(a, b)$, by*

$$\max(a, b) = \begin{cases} b & \text{if} \quad a \leq b \\ a & \text{if} \quad a > b \end{cases}$$

We define the **minimum** *of a, b, which we denote by $\min(a, b)$, by*

$$\min(a, b) = \begin{cases} a & \text{if} \quad a \leq b \\ b & \text{if} \quad a > b \end{cases}$$

Prove the statements below:

(a) *Suppose $a, b \in \mathbb{R}$. Then $\min(a, b) \leq a \leq \max(a, b)$ and $\min(a, b) \leq b \leq \max(a, b)$.*

(b) *Suppose $a, b \in \mathbb{R}$. Then $\max(a, b) = \dfrac{a + b + |a - b|}{2}$ and $\min(a, b) = \dfrac{a + b - |a - b|}{2}$.*

(c) *Suppose $a, b \in \mathbb{R}$. Then $a + b = \max(a, b) + \min(b, a)$ and $|a - b| = \max(a, b) - \min(a, b)$.*

(d) *Suppose $a, b \in \mathbb{R}$. Then $\max(a, b) = \max(b, a)$ and $\min(a, b) = \min(b, a)$.*

(e) *Suppose $a, b \in \mathbb{R}$. Then $\max(-a, -b) = -\min(a, b)$ and $\min(-a, -b) = -\max(a, b)$.*

(f)$^\diamond$ *Suppose $a, b, c \in \mathbb{R}$. Then $\max(\max(a, b), c) = \max(a, \max(b, c))$ and $\min(\min(a, b), c) = \min(a, \min(b, c))$.*

(g)$^\diamond$ *Suppose $a, b, c \in \mathbb{R}$. Then $\min(\max(a, b), \max(b, c), \max(a, c)) = \max(\min(a, b), \min(b, c), \min(c, a))$.*

23. (a) Prove the statements below:

i. *Suppose $a \in \mathbb{R}$. Then $|a| = |-a|$.*

ii. *Suppose $b, c \in \mathbb{R}$. Then $-b \leq c \leq b$ iff $|c| \leq b$.*

(b) Apply the results above to prove the statement below:

- *Suppose $x, y \in \mathbb{R}$. Then $|x + y| \leq |x| + |y|$.*

24. Let ε be a positive real number. Let x, y, x_0, y_0 be real numbers. Prove the statements below:

(a)$^\diamond$ *Suppose $|x - x_0| < \frac{\varepsilon}{2}$ and $|y - y_0| < \frac{\varepsilon}{2}$. Then $|(x + y) - (x_0 + y_0)| < \varepsilon$ and $|(x - y) + (x_0 - y_0)| < \varepsilon$.*

(b)$^\clubsuit$ *Suppose $|x - x_0| < \min(\dfrac{\varepsilon}{2|y_0| + 1}, 1)$ and $|y - y_0| < \dfrac{\varepsilon}{2|x_0| + 1}$. Then $|xy - x_0 y_0| < \varepsilon$.*

(c)$^\clubsuit$ *Suppose $y_0 \neq 0$ and $|y - y_0| < \min(\dfrac{|y_0|}{2}, \dfrac{\varepsilon |y_0|^2}{2})$. Then $y \neq 0$ and $\left| \dfrac{1}{y} - \dfrac{1}{y_0} \right| < \varepsilon$.*

25.$^\diamondsuit$ Let c, ε be positive real numbers. Define $\delta = \min(1, \dfrac{\varepsilon}{1+3c+3c^2})$.

(a) Prove that $\delta > 0$ and $\delta \leq 1$.

(b) Let x be a real number. Suppose $|x - c| < \delta$.

i. Prove that $|x^2 + cx + c^2| \leq 1 + 3c + 3c^2$.

ii. Hence, or otherwise, deduce that $|x^3 - c^3| < \varepsilon$.

Remark. This is what we have verified overall: *For any $c > 0$, for any $\varepsilon > 0$, there exists some $\delta > 0$, (namely, $\delta = \min(1, \dfrac{\varepsilon}{1+3c+3c^2})$) such that for any $x \in \mathbb{R}$, if $|x-c| < \delta$ then $|x^3 - c^3| < \varepsilon$.* Hence we have argued for the continuity of the function t^3 at every positive value of t.

26.$^\diamondsuit$ Let c, ε be positive real numbers. Define $\delta = \min(\dfrac{\varepsilon c^2}{2}, \dfrac{c}{2})$.

(a) Prove that $\delta > 0$ and $\delta \leq \dfrac{c^2}{2}$.

(b) Let x be a real number. Suppose $|x - c| < \delta$.

i. Prove that $x > \dfrac{c}{2}$.

ii. Hence, or otherwise, deduce that $\left|\dfrac{1}{x} - \dfrac{1}{c}\right| < \varepsilon$.

Remark. This is what we have verified overall: *For any $c > 0$, for any $\varepsilon > 0$, there exists some $\delta > 0$, (namely, $\delta = \min(\dfrac{\varepsilon c^2}{2}, \dfrac{c}{2})$) such that for any $x \in \mathbb{R}$, if $|x-c| < \delta$ then $|1/x - 1/c| < \varepsilon$.* Hence we have argued for the continuity of the function $1/t$ at every positive value of t.

27.$^\diamondsuit$ Let c, ε be positive real numbers. Define $\delta = \min(\varepsilon\sqrt{c}, \dfrac{c}{2})$.

(a) Prove that $\delta > 0$ and $\delta \leq \dfrac{c^2}{2}$.

(b) Suppose $|x - c| < \delta$.

i. Prove that $x > \dfrac{c}{2}$.

ii. Hence, or otherwise, deduce that $|\sqrt{x} - \sqrt{c}| < \varepsilon$.

Remark. This is what we have verified overall: *For any $c > 0$, for any $\varepsilon > 0$, there exists some $\delta > 0$, (namely, $\delta = \min(\varepsilon\sqrt{c}, \dfrac{c}{2})$) such that for any $x \in \mathbb{R}$, if $|x-c| < \delta$ then $|\sqrt{x} - \sqrt{c}| < \varepsilon$.* Hence we have argued for the continuity of the function $\sqrt{t}$ at every positive value of t.

28. (a) Prove the statement below:

- *Suppose $u, v \in \mathbb{R}$. Then $|u| \leq \sqrt{u^2+v^2}$ and $|u+v| \leq 2\sqrt{u^2+v^2}$.*

(b)$^\diamondsuit$ Let a, b be real numbers, and ε be a positive real number. Define $\delta = \dfrac{\varepsilon}{2}$.

Suppose $\sqrt{(x-a)^2 + (y-b)^2} < \delta$. Prove that $|(x+y) - (a+b)| < \varepsilon$.

Remark. This is what we have verified overall: *For any $a, b \in \mathbb{R}$, for any $\varepsilon > 0$, there exists some $\delta > 0$, (namely, $\delta = \dfrac{\varepsilon}{2}$) such that for any $x, y \in \mathbb{R}$, if $\sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|(x+y) - (a+b)| < \varepsilon$.* Hence we have argued for the continuity of the function $s + t$ at every point (s, t) on the plane $\mathbb{R}^2$.

(c)$^\clubsuit$ Let a, b be real numbers, and ε be a positive real number. Define $\delta = \min(\dfrac{\varepsilon}{|a|+|b|+1}, 1)$.

Suppose $\sqrt{(x-a)^2 + (y-b)^2} < \delta$. Prove that $|xy - ab| < \varepsilon$.

Remark. This is what we have verified overall: *For any $a, b \in \mathbb{R}$, for any $\varepsilon > 0$, there exists some $\delta > 0$, (namely, $\delta = \min(\dfrac{\varepsilon}{|a|+|b|+1}, 1)$) such that for any $x, y \in \mathbb{R}$, if $\sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|xy - ab| < \varepsilon$.* Hence we have argued for the continuity of the function $s + t$ at every point (s, t) on the plane $\mathbb{R}^2$.

29. *Take for granted the validity of the statement below, (which is a special case of* **Bernoulli's Inequality***):*

- Let $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}\backslash\{0,1\}$. Suppose $-1 < \alpha < 0$ or $\alpha > 0$. Then $(1+\alpha)^m > 1 + m\alpha$.

(a) Let $a \in \mathbb{R}$ and $n \in \mathbb{N}\backslash\{0\}$. Suppose $a > 1$.

i.$^\diamondsuit$ Applying Bernoulli's Inequlaity, or otherwise, prove that $a^{n+1} - 1 < (n+1)(a-1)a^n$.

ii. Hence deduce that $\dfrac{a^{n+1}-1}{n+1} > \dfrac{a^n - 1}{n}$.

(b)$^\diamondsuit$ Hence prove the statement below:

- *Let $a \in \mathbb{R}$ and $k, \ell \in \mathbb{N}\backslash\{0\}$. Suppose $a > 1$, and $k > \ell$. Then $\dfrac{a^k - 1}{k} > \dfrac{a^\ell - 1}{\ell}$.*

(c)$^\clubsuit$ Hence prove the statements below:

i. *Let $b \in \mathbb{R}$ and $r \in \mathbb{Q}$. Suppose $b > 1$ and $r > 1$. Then $\dfrac{b^r - 1}{r} > b - 1$.*

ii. *Let $\beta \in \mathbb{R}$ and $r \in \mathbb{Q}$. Suppose $\beta > 0$ and $r > 1$. Then $(1+\beta)^r > 1 + r\beta$.*

iii. *Let $c \in \mathbb{R}$ and $s, t \in \mathbb{Q}$. Suppose $c > 1$ and $s > t > 1$. Then $\dfrac{c^s - 1}{s} > \dfrac{c^t - 1}{t}$.*

Remark. Modifying the arguments with which you work out the arguments in this question, try to prove the statements below as well:

(A) *Let $\beta \in \mathbb{R}$ and $r \in \mathbb{Q}$. Suppose $-1 < \beta < 0$ and $r > 1$. Then $(1+\beta)^r > 1 + r\beta$.*

(B) *Let $\beta \in \mathbb{R}$ and $r \in \mathbb{Q}$. Suppose $-1 < \beta < 0$ or $\beta > 0$ and $0 < r < 1$. Then $(1+\beta)^r < 1 + r\beta$.*

(C) *Let $c \in \mathbb{R}$ and $s, t \in \mathbb{Q}$. Suppose $0 < c < 1$ and $s > t > 1$. Then $\dfrac{1 - c^s}{s} < \dfrac{1 - c^t}{t}$.*

30. Let $c \in \mathbb{R}$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = -x^2 + c$ for any $x \in \mathbb{R}$.

(a) Verify according to definition that the function f attains absolute maximum at 0, with absolute maximum value c.

(b) Verify according to definition that f is strictly increasing on $(-\infty, 0]$.

(c) Verify according to definition that f is strictly decreasing on $[0, +\infty)$.

(d)$^\diamondsuit$ Verify according to definition that f is strictly concave on $\mathbb{R}$.

31. Let $b \in \mathbb{R}$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^2 + bx$ for any $x \in \mathbb{R}$.

(a) Verify according to definition that the function f attains absolute minimum at $-b/2$, with absolute minimum value $-b^2/4$.

(b) Verify according to definition that f is strictly decreasing on $(-\infty, -b/2]$.

(c) Verify according to definition that f is strictly increasing on $[-b/2, +\infty)$.

(d)$^\diamondsuit$ Verify according to definition that f is strictly convex on $\mathbb{R}$.

32. (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^3$ for any $x \in \mathbb{R}$. Verify that f is strictly increasing on $\mathbb{R}$, according to definition.

(b) Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $g(x) = x^3 + x$ for any $x \in \mathbb{R}$. Verify that g is strictly increasing on $\mathbb{R}$, according to definition.

(c) Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $h(x) = x^3 - 3x$ for any $x \in \mathbb{R}$. Verify that h is strictly increasing on $(-\infty, -1]$ and on $[1, +\infty)$, and that h is strictly decreasing on $[-1, 1]$, according to definition.

33. *In this question you are assumed to be familiar with one-variable calculus.*

Take for granted the validity of the statement (MVT), known as **Mean-Value Theorem**:

(MVT) *Let $a, b \in \mathbb{R}$, with $a < b$, and f be a function defined on $[a, b]$.*
Suppose f satisfies all the conditions below:

(C) *f is continuous on $[a, b]$.* (D) *f is differentiable on (a, b).*

Then there exists some $\zeta \in (a, b)$ such that $f(b) - f(a) = (b - a)f'(\zeta)$.

(a)$^\diamondsuit$ Apply the Mean-Value Theorem to deduce the statement (SI) below, (which relates strict monotonicity to the sign of the first derivative):

(SI) *Let f be a real-valued function defined on some open interval I in $\mathbb{R}$.*
Suppose f satisfies all the conditions:

(C) f *is continuous on* I. (D) f *is differentiable on* I. (P) $f'(x) > 0$ *for any* $x \in I$.

Then f *is strictly increasing on* I.

Remark. Formulate and prove the analogous result for strictly decreasing functions. This pair of results provides some useful tools for proving inequalities. It also constitutes the theoretical foundation for the 'First Derivative Test' for checking relative extrema. (Refer to the material in your *calculus* course.)

(b)♣ Apply the Mean-Value Theorem, the statement (SI), to prove the statement (SV) (which relates strict convexity/concavity to the sign of the second derivative):

(SV) *Let* f *be a real-valued function defined on some open interval* I *in* $\mathbb{R}$.
Suppose f *satisfies all the conditions:*

(D2) f *is twice differentiable on* I. (P2) $f''(x) > 0$ *for any* $x \in I$.

Then f *is strictly convex on* I.

Remark. Formulate and prove the analogous result for strictly concave functions. This pair of results constitutes the theoretical foundation for the 'Second Derivative Test' for checking relative extrema. (Refer to the material in your *calculus* course.)

34. Let $f : (0, 10) \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \sqrt{\dfrac{1000 - x^3}{3x}}$ for any $x \in (0, 10)$.

(a) Verify that $f'(x) = A\left(\dfrac{f(x)}{x} + \dfrac{x}{f(x)}\right)$ for any $x \in (0, 6)$. Here A is a constant whose value you have to determine explicitly.

(b) Hence, or otherwise, prove that f is strictly monotonic on $(0, 6)$. Is f strictly increasing on $(0, 6)$ or strictly decreasing on $(0, 6)$?

35. (a) Let $f : (0, +\infty) \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{e^x}{x^e}$ for any $x \in (0, +\infty)$. By using calculus, or otherwise, prove that that f is strictly decreasing on $(0, e]$ and f is strictly increasing on $[e, +\infty)$.

(b) Hence, or otherwise, prove that $e^\pi > \pi^e$.

36.◊ Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be the functions defined respectively by $f(x) = xe^{-x^2}$, $g(x) = x - xe^{-x^2}$ for any $x \in \mathbb{R}$.

(a) By using calculus, or otherwise, prove that f is strictly increasing on $[-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}]$.

(b) Prove that g is strictly increasing on $[0, +\infty)$.

(c) i. Prove that $ae^{b^2} < be^{a^2}$ whenever $-\dfrac{1}{\sqrt{2}} < a < b < \dfrac{1}{\sqrt{2}}$.

ii. Prove that $0 < \dfrac{be^{a^2} - ae^{b^2}}{b - a} < e^{a^2+b^2}$ whenever $0 < a < b < \dfrac{1}{\sqrt{2}}$.

37.◊ By using calculus, or otherwise, prove the statements below:

(a) $1 + 2x - 2x^2 \leq \sqrt{1 + 4x} \leq 1 + 2x$ *whenever* $x \geq 0$.

(b) *Suppose* p *be a positive integer. Then* $\dfrac{x^{p+1} - 1}{p + 1} \geq \dfrac{x^p - 1}{p}$ *whenever* $x > 0$.

(c) *Suppose* α *is a rational number greater than* 1. *Then* $\dfrac{1}{2^{\alpha-1}} \leq x^\alpha + (1 - x)^\alpha \leq 1$ *whenever* $0 \leq x \leq 1$.

(d) $\sin(x) > x\cos(x)$ *whenever* $0 < x < \pi$.

(e) $x\sin(x) + \cos(x) > 1 + \dfrac{1}{2}x^2\cos(x)$ *whenever* $0 < x < \pi$.