MATH1050 Exercise 2 (Answers and selected solution)

1. **Solution.**

   Let a, b, c be numbers, with $a \neq 0$. Let α be a number. Let $f(x)$ be the quadratic polynomial given by $ax^2 + bx + c$.

   (a) Suppose α is a root of $f(x)$. Let $\beta = -\dfrac{b}{a} - \alpha$.

   i. We have $0 = f(\alpha) = a\alpha^2 + b\alpha + c$. Then $c = -a\alpha^2 - b\alpha$.
   Therefore, as polynomials, $f(x) = ax^2 + bx + c = ax^2 + bx - a\alpha^2 - b\alpha^2 = a(x^2 - \alpha^2) + b(x - \alpha) = (x - \alpha)[a(x + \alpha) + b] = a(x - \alpha)(x + \alpha + \dfrac{b}{a}) = a(x - \alpha)(x - \beta)$.

   ii. We have $f(\beta) = a(\beta - \alpha)(\beta - \beta) = 0$. Then β is a root of $f(x)$.

   iii. As polynomials, $ax^2 + bx + c = f(x) = a(x - \alpha)(x - \beta) = ax^2 - a(\alpha + \beta)x + a\alpha\beta$. By comparing coefficents, we have $c = a\alpha\beta$. Then $\alpha\beta = \dfrac{c}{a}$.

   (b) Define $\Delta_f = b^2 - 4ac$.

   i. As polynomials,

   $$\begin{aligned} f(x) &= ax^2 + 2a \cdot \frac{b}{2a}x + a \cdot \frac{b^2}{4a^2} - a \cdot \frac{b^2}{4a^2} + a \cdot \frac{4ac}{a^2} = a\left[\left(x^2 + 2 \cdot \frac{b}{2a}x + \frac{b^2}{4a^2}\right) - \left(\frac{b^2 - 4ac}{4a^2}\right)\right] \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta_f}{4a^2}\right]. \end{aligned}$$

   ii. Suppose a, b, c are real numbers.

   A. Suppose $\Delta_f \geq 0$. Define $\alpha_\pm = \dfrac{-b \pm \sqrt{\Delta_f}}{2a}$ respectively.
   Note that $f(\alpha_+) = a\left[\left(\alpha_+ + \dfrac{b}{2a}\right)^2 - \dfrac{\Delta_f}{4a^2}\right] = \cdots = 0$.
   Then $f(\alpha_+)$ is a root of $f(x)$.
   We have $\alpha_+ + \alpha_- = -\dfrac{b}{a}$. Then $\alpha_- = -\dfrac{b}{a} - \alpha_+$.
   By the result in part (a), α_- is also a root of $f(x)$, and $f(x) = a(x - \alpha_+)(x - \alpha_-)$ as polynomials.

   B. Suppose suppose $\Delta_f < 0$ instead. Define $\zeta = \dfrac{-b + i\sqrt{-\Delta_f}}{2a}$. Further define $\bar{\zeta} = \dfrac{-b - i\sqrt{-\Delta_f}}{2a}$.
   Note that $f(\zeta) = a\left[\left(\zeta + \dfrac{b}{2a}\right)^2 - \dfrac{\Delta_f}{4a^2}\right] = \cdots = 0$.
   Then $f(\zeta)$ is a root of $f(x)$.
   We have $\zeta + \bar{\zeta} = -\dfrac{b}{a}$. Then $\bar{\zeta} = -\dfrac{b}{a} - \zeta$.
   By the result in part (a), $\bar{\zeta}$ is also a root of $f(x)$, and $f(x) = a(x - \zeta)(x - \bar{\zeta})$ as polynomials.

   (c) Now we no longer suppose '*a, b, c are real numbers*'.

   i. Suppose $\Delta_f \neq 0$, and σ is a square root of $\dfrac{\Delta_f}{4a^2}$ in $\mathbb{C}$. Define $\alpha_\pm = -\dfrac{b}{2a} \pm \sigma$ respectively.
   Note that $f(\alpha_+) = a\left[\left(\alpha_+ + \dfrac{b}{2a}\right)^2 - \dfrac{\Delta_f}{4a^2}\right] = \cdots = 0$.
   Then $f(\alpha_+)$ is a root of $f(x)$.
   We have $\alpha_+ + \alpha_- = -\dfrac{b}{a}$. Then $\alpha_- = -\dfrac{b}{a} - \alpha_+$.
   By the result in part (a), α_- is also a root of $f(x)$, and $f(x) = a(x - \alpha_+)(x - \alpha_-)$ as polynomials.

   ii. Now suppose $\Delta_f = 0$ instead.
   As polynomials, $f(x) = a\left[\left(x + \dfrac{b}{2a}\right)^2 - \dfrac{\Delta_f}{4a^2}\right] = a\left(x + \dfrac{b}{2a}\right)^2$.

2. **Solution.**

Let a, b, c, r be numbers, with $a \neq 0$ and $c \neq 0$ and $r \neq 0$. Let $f(x)$ be the quadratic polynomial given by $f(x) = ax^2 + bx + c$. Suppose α, β are the roots of $f(x)$. Further suppose $\alpha = r\beta$.

Since α, β are the roots of $f(x)$, we have

$$\begin{cases} \alpha + \beta &= -b/a \\ \alpha\beta &= c/a \end{cases}$$

Then we have $(r+1)\beta = \alpha + \beta = -\dfrac{b}{a}$ and $r\beta^2 = \alpha\beta = \dfrac{c}{a}$.

Therefore $\dfrac{b^2}{a^2} = (r+1)^2\beta^2 = \dfrac{(r+1)^2}{r} \cdot r\beta^2 = \dfrac{(r+1)^2}{r} \cdot \dfrac{c}{a}$.

Hence $rb^2 = (r+1)^2 ac$.

3. **Solution.**

(a) We proceed to solve the inequality $(\star)$:

$$\begin{array}{rcl} x^2 - 3x & < & 10 \quad \text{——} \quad (\star) \\ x^2 - 3x - 10 & < & 0 \\ (x+2)(x-5) & < & 0 \\ (x+2<0 \text{ and } x-5>0) & \text{or} & (x+2>0 \text{ and } x-5<0) \\ \underbrace{(x<-2 \text{ and } x>5)}_{\text{(rejected)}} & \text{or} & -2<x<5 \\ -2<x & < & 5 \end{array}$$

(Every line is logically equivalent to the next. No checking of solution is needed.)

The solution of the inequality $(\star)$ is $-2 < x < 5$.

(b) We proceed to solve the system of inequalities $(\star)$:

$$\begin{array}{rcl} (x+1)(x-6) \geq 8 & \text{and} & 3x - 1 \geq 5 \quad \text{——} \quad (\star) \\ x^2 - 5x - 14 \geq 0 & \text{and} & x \geq 2 \\ (x+2)(x-7) \geq 0 & \text{and} & x \geq 2 \\ (x \leq -2 \text{ or } x \geq 7) & \text{and} & x \geq 2 \\ \underbrace{(x \leq -2 \text{ and } x \leq 2)}_{\text{(rejected)}} & \text{or} & (x \geq 7 \text{ and } x \geq 2) \\ x & \geq & 7 \end{array}$$

(Every line is logically equivalent to the next. No checking of solution is needed.)

The solution of the system of inequalities $(\star)$ is $x \geq 7$.

(c) We proceed to solve the system of inequalities $(\star)$:

$$\begin{array}{rcl} (x+1)^2 > 16 & \text{or} & 2x + 5 > 7 \quad \text{——} \quad (\star) \\ (x+1 < -4 \text{ or } x+1 > 4) & \text{or} & x > 1 \\ x < -5 \text{ or } x > 3 & \text{or} & x > 1 \\ x < -5 & \text{or} & x > 1 \end{array}$$

(Every line is logically equivalent to the next. No checking of solution is needed.)

The solution of the system of inequalities $(\star)$ is $x < -5$ or $x > 1$.

(d) We proceed to solve the inequality $(\star)$:

$$\begin{array}{rcl} (x-1)(x-2)(x-3) & \geq & 0 \quad \text{——} \quad (\star) \\ ((x-1)(x-2) \leq 0 \text{ and } x-3 \leq 0) & \text{or} & ((x-1)(x-2) \geq 0 \text{ and } x-3 \geq 0) \\ (1 \leq x \leq 2 \text{ and } x \leq 3) & \text{or} & ((x \leq 1 \text{ or } x \geq 2) \text{ and } x \geq 3) \\ 1 \leq x \leq 2 & \text{or} & x \geq 3 \end{array}$$

(Every line is logically equivalent to the next. No checking of solution is needed.)

The solution of the inequality $(\star)$ is $1 \leq x \leq 2$ or $x \geq 3$.

(e) We proceed to solve the inequality ($\star$):

$$\begin{array}{rcll} \frac{2}{3-x} & \leq & 1 & \text{—— } (\star) \\ 2(3-x) & \leq & (3-x)^2 & \text{and } x \neq 3 \\ (x-3)^2 + 2(x-3) & \geq & 0 & \text{and } x \neq 3 \\ (x-1)(x-3) & \geq & 0 & \text{and } x \neq 3 \\ (x \leq 1 & \text{or} & x \geq 3) & \text{and } x \neq 3 \\ x \leq 1 & \text{or} & x > 3 & \end{array}$$

(Every line is logically equivalent to the next. No checking of solution is needed.)

The solution of the inequality ($\star$) is $x \leq 1$ or $x > 3$.

(f) We proceed to solve the inequality ($\star$):

$$\begin{array}{rcll} 2x - \frac{3}{x} & \geq & 1 & \text{—— } (\star) \\ 2x^3 - 3x & \geq & x^2 & \text{and } x \neq 0 \\ 2x^3 - x^2 - 3x & \geq & 0 & \text{and } x \neq 0 \\ x(x+1)(2x-3) & \geq & 0 & \text{and } x \neq 0 \\ (-1 \leq x \leq 0 & \text{or} & x \geq 1.5) & \text{and } x \neq 0 \\ -1 \leq x < 0 & \text{or} & x \geq 1.5 & \end{array}$$

(Every line is logically equivalent to the next. No checking of solution is needed.)

The solution of the inequality ($\star$) is $-1 \leq x < 0$ or $x \geq 1.5$.

(g) We proceed to solve the inequality ($\star$):

$$\begin{array}{rcll} \frac{x^2-1}{x^2-4} & \leq & -2 & \text{—— } (\star) \\ (x^2-1)(x^2-4) & \leq & -2(x^2-4)^2 & \text{and } x \neq -2 \text{ and } x \neq 2 \\ (x^2-4)[(x^2-1) + 2(x^2-4)] & \leq & 0 & \text{and } x \neq -2 \text{ and } x \neq 2 \\ (x^2-4)(3x^2-9) & \leq & 0 & \text{and } x \neq -2 \text{ and } x \neq 2 \\ 3(x+2)(x+\sqrt{3})(x-\sqrt{3})(x-2) & \leq & 0 & \text{and } x \neq -2 \text{ and } x \neq 2 \\ (-2 \leq x \leq -\sqrt{3} \text{ or } \sqrt{3} \leq x \leq 2) & & & \text{and } x \neq -2 \text{ and } x \neq 2 \\ -2 < x \leq -\sqrt{3} \text{ or } \sqrt{3} \leq x < 2 & & & \end{array}$$

(Every line is logically equivalent to the next. No checking of solution is needed.)

The solution of the inequality ($\star$) is $-2 < x \leq -\sqrt{3}$ or $\sqrt{3} \leq x < 2$.

(h) We proceed to solve the inequality ($\star$):

$$\begin{array}{rcl} |x^2 - 5x| & < & 6 \quad \text{—— } (\star) \\ x^2 - 5x > -6 & \text{and} & x^2 - 5x < 6 \\ x^2 - 5x + 6 > 0 & \text{and} & x^2 - 5x - 6 < 0 \\ (x-2)(x-3) > 0 & \text{and} & (x+1)(x-6) < 0 \\ (x < 2 \text{ or } x > 3) & \text{and} & -1 < x < 6 \\ (x < 2 \text{ and } -1 < x < 6) & \text{or} & (x > 3 \text{ and } -1 < x < 6) \\ -1 < x < 2 & \text{or} & 3 < x < 6 \end{array}$$

(Every line is logically equivalent to the next. No checking of solution is needed.)

The solution of the inequality ($\star$) is $-1 < x < 2$ or $3 < x < 6$.

(i) We proceed to solve the inequality $(\star)$:

$$\begin{aligned}
\left|\frac{3x+11}{x+2}\right| &< 2 \quad \text{—} \quad (\star) \\
\frac{|3x+11|}{|x+2|} &< 2 \\
|3x+11| &< 2|x+2| \quad \text{and } x \neq -2 \\
(3x+11)^2 &< 4(x+2)^2 \quad \text{and } x \neq -2 \\
9x^2+66x+121 &< 4x^2+16x+16 \quad \text{and } x \neq -2 \\
x^2+10x+21 &< 0 \quad \text{and } x \neq -2 \\
(x+3)(x+7) &< 0 \quad \text{and } x \neq -2 \\
-7 < x &< -3 \quad \text{and } x \neq -2 \\
-7 < x &< -3
\end{aligned}$$

(Every line is logically equivalent to the next. No checking of solution is needed.)

The solution of the inequality $(\star)$ is $-7 < x < -3$.

(j) We proceed to solve the inequality $(\star)$:

$$\begin{aligned}
\big|\ |x|-4\ \big| \quad &> \quad 3 \quad \text{—} \quad (\star) \\
|x|-4 < -3 \quad &\text{or} \quad |x|-4 > 3 \\
|x| < 1 \quad &\text{or} \quad |x| > 7 \\
-1 < x < 1 \quad &\text{or} \quad x < -7 \quad \text{or} \quad x > 7
\end{aligned}$$

(Every line is logically equivalent to the next. No checking of solution is needed.)

The solution of the inequality $(\star)$ is $-1 < x < 1$ or $x < -7$ or $x > 7$.

(k) We proceed to solve the inequality $(\star)$:

$$\begin{aligned}
|x^2-3| &\leq 2|x| \quad \text{—} \quad (\star) \\
(x^2-3)^2 &\leq 4x^2 \\
x^4-6x^2+9 &\leq 4x^2 \\
x^4-10x^2+9 &\leq 0 \\
(x^2-1)(x^2-9) &\leq 0 \\
(x+3)(x+1)(x-1)(x-3) &\leq 0 \\
-3 \leq x \leq -1 \quad &\text{or} \quad 1 \leq x \leq 3
\end{aligned}$$

(Every line is logically equivalent to the next. No checking of solution is needed.)

The solution of the inequality $(\star)$ is $-3 \leq x \leq -1$ or $1 \leq x \leq 3$.

(l) We proceed to solve the inequality $(\star)$:

$$\begin{aligned}
|2x+1| \quad &< \quad 3x-2 \quad \text{—} \quad (\star) \\
0 \leq 2x+1 < 3x-2 \quad &\text{or} \quad 0 \leq -2x-1 < 3x-2 \\
(x \geq -0.5 \text{ and } x > 3) \quad &\text{or} \quad \underbrace{(x \leq -0.5 \text{ and } x > 0.2)}_{\text{(rejected)}} \\
x > 3 &
\end{aligned}$$

(Every line is logically equivalent to the next. No checking of solution is needed.)

The solution of the inequality $(\star)$ is $x > 3$.

4. **Solution.**

Let p be a real number. Let $f(x)$ be the quadratic polynomial given by $f(x) = x^2 + (p+1)x + (p-1)$. Suppose α, β are the roots of $f(x)$.

(a) The discriminant Δ_f of the polynomial $f(x)$ is given by $\Delta_f = (p+1)^2 - 4 \cdot 1 \cdot (p-1)$.

We have $\Delta_f = (p+1)^2 - 4 \cdot 1 \cdot (p-1) = p^2 - 2p + 5 = (p-1)^2 + 4 \geq 4 > 0$.

Therefore the roots of $f(x)$, which are α, β, are real and distinct.

(b) $(\alpha-2)(\beta-2) = \alpha\beta - 2(\alpha+\beta) + 4 = (p-1) - 2[-(p+1)] + 4 = 3p+5$.

(c) Suppose $\beta < 2 < \alpha$.

i. Since $\beta < 2 < \alpha$, we have $\beta - 2 < 0$ and $\alpha - 2 > 0$. Then $3p+5 = (\alpha-2)(\beta-2) < 0$. Therefore $p < -\frac{5}{3}$.

ii. Further suppose $(\alpha-\beta)^2 < 20$.
Note that $(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta = \Delta_f = (p-1)^2 + 4$.
Since $(\alpha-\beta)^2 < 20$, we have $(p-1)^2 + 4 < 20$. Then $(p-1)^2 < 16$. Therefore $-4 < p-1 < 4$. Hence $-5 < p < 3$.
Recall that $p < -\frac{5}{3}$. Therefore $p < -\frac{5}{3}$ and $-3 < p < 5$.
Hence $-3 < p < -\frac{5}{3}$.

5. ——

6. **Answer.**

(a) (I) Suppose $x+y > 1$ and $x > y$
(II) Since
(III) $x > y$
(IV) $(x-y)(x+y) - (x-y) = (x-y)(x+y-1)$
(V) $x^2 - y^2 > x - y$

(b) (I) Let $x, y \in \mathbb{R}$. Suppose $x > 0$ and $y > 0$.
(II) $(x+y)(x^2-xy+y^2) - xy(x+y) = (x+y)(x^2-2xy+y^2) = (x+y)(x-y)^2 \geq 0$

7. **Answer.**

(I) Suppose $y > x > 0$ and $z > -y$
(II) $z > -y$
(III) > 0
(IV) Suppose
(V) $zy > zx$
(VI) $\frac{x+z}{y+z} - \frac{x}{y} = \frac{(x+z)y - x(y+z)}{y(y+z)} > 0$
(VII) Suppose $\frac{x+z}{y+z} > \frac{x}{y}$
(VIII) $\frac{x+z}{y+z} \cdot y(y+z) > \frac{x}{y} \cdot y(y+z)$
(IX) Therefore $z(y-x) = zy - zx > 0$. Since $y > x$, we have $y - x > 0$.
(X) $\frac{x+z}{y+z} > \frac{x}{y}$ iff $z > 0$

8. *Hint.* The key is this 'factorization':

$$\left(x^m + \frac{1}{x^m}\right) - \left(x^n + \frac{1}{x^n}\right) = \frac{(x^m - x^n)(x^{m+n} - 1)}{x^{m+n}}.$$

9. **Solution.**

(a) Let $u, v, x, y \in \mathbb{R}$.
We have $(ux+vy)^2 = u^2x^2 + 2uxvy + v^2y^2$.
Also, we have $(u^2+v^2)(x^2+y^2) = u^2x^2 + u^2y^2 + v^2x^2 + v^2y^2$.
Then $(u^2+v^2)(x^2+y^2) - (ux+vy)^2 = u^2y^2 + v^2x^2 - 2uxvy = (uy-vx)^2 \geq 0$.
Therefore $(ux+vy)^2 \leq (u^2+v^2)(x^2+y^2)$.

(b) Let s, t be positive real numbers. $\sqrt{s}, \sqrt{t}$ are well-defined as real numbers, and $s = (\sqrt{s})^2$, $t = (\sqrt{t})^2$.

$$(s+t)\left(\frac{1}{s} + \frac{1}{t}\right) = [(\sqrt{s})^2) + (\sqrt{t})^2]\left[\left(\frac{1}{\sqrt{s}}\right)^2 + \left(\frac{1}{\sqrt{t}}\right)^2\right] \geq \left(\sqrt{s} \cdot \frac{1}{\sqrt{s}} + \sqrt{t} \cdot \frac{1}{\sqrt{t}}\right)^2 = (1+1)^2 = 4.$$

10. (a) *Hint.* Repeatedly apply the inequality for real numbers '$u^2 + v^2 \geq 2uv$'.

    (b) *Hint.* Take $a = r$, $b = s$, $c = t$ $d = \dfrac{r+s+t}{3}$. An alternative is to take $a = r$, $b = s$, $c = t$ $d = \sqrt[3]{rst}$.

11. **Solution.**

    Let c, ε be positive real numbers. Define $\delta = \sqrt{c^2 + \varepsilon} - c$.

    (a) Note that $c^2 + \varepsilon > c^2 \geq 0$. Then $\sqrt{c^2 + \varepsilon} > c$. Therefore $\delta = \sqrt{c^2 + \varepsilon} - c > 0$.

    (b) Let x be a real number. Suppose $|x - c| < \delta$.

    i. We have $|x + c| = |(x - c) + 2c| \leq |x - c| + 2c \leq \delta + 2c = \sqrt{c^2 + \varepsilon} + c$.

    ii. We have $|x^2 - c^2| = |x - c| \cdot |x + c| < \delta \cdot (\sqrt{c^2 + \varepsilon} + c) = (\sqrt{c^2 + \varepsilon} - c)(\sqrt{c^2 + \varepsilon} + c) = c^2 + \varepsilon - c^2 = \varepsilon$.

12. ——